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# HERMITIAN BANACH *-ALGEBRAS

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We establish several new characterizations of hermitian Banach *-algebras among all Banach *-algebras. In particular we show that a Banach *-algebra is hermitian if and only if its Gelfand-Naimark pseudo-norm is a $Q$-pseudo-norm. Much of the theory of hermitian Banach *-algebras can be derived directly from this fact (cf. [1]).

We establish our terminology and notation. A Banach *-algebra is a Banach algebra over the complex numbers together with a fixed involution denoted by *. No connection between the norm and involution is postulated. A Banach *-algebra is called hermitian iff the spectrum of each element $h = h^*$ in $\mathfrak{A}$ is contained in the real line.

A linear functional $\omega$ on a Banach *-algebra $\mathfrak{A}$ is called positive iff $\omega(a^*a) \geqq 0$ for all $a \in \mathfrak{A}$. The left kernel $\mathfrak{A}_\omega$ of a positive linear functional $\omega$ on $\mathfrak{A}$ is the left ideal $\{a \in \mathfrak{A} : \omega(a^*a) = 0\}$. If $\omega$ is a positive linear functional on $\mathfrak{A}$ and 1 is the minimum value of $B$ such that

$$|\omega(a)|^2 \leqq B\omega(a^*a) \qquad \forall a \in \mathfrak{A},$$

then $\omega$ is called a state.

A pure state is an extreme point of the convex set, $\mathfrak{A}_s^+$, of states. A *-representation of a Banach *-algebra $\mathfrak{A}$ is an algebra homomorphism $T$ of $\mathfrak{A}$ into the algebra $[\mathfrak{H}]$ of all bounded linear operators on some Hilbert space $\mathfrak{H}$, such that $T_{a^*} = (T_a)^*$ for all $a \in \mathfrak{A}$. A $B^*$-pseudo-norm on a Banach *-algebra $\mathfrak{A}$ is a submultiplicative pseudo-norm (=seminorm) $\tau$ on $\mathfrak{A}$ such that $\tau(a^*a) = \tau(a)^2$ for all $a \in \mathfrak{A}$. On any Banach *-algebra $\mathfrak{A}$ there is a maximum $B^*$-pseudo-norm $\gamma$ which satisfies

$$\begin{aligned}\gamma(a) &= \sup\{\|T_a\| : T \text{ is a *-representation of } \mathfrak{A}\}\\ &= \sup\{\omega(a^*a)^{1/2} : \omega \in \mathfrak{A}_s^+\}.\end{aligned}$$

This pseudo-norm $\gamma$ is called the Gelfand-Naimark pseudo-norm.

A pseudo-norm $\tau$ on an algebra $\mathfrak{A}$ is called a $Q$-pseudo-norm iff it is submultiplicative and satisfies any of the following equivalent conditions: (1) the set $\mathfrak{A}_{qG}$ of quasi-regular elements in $\mathfrak{A}$ is $\tau$-open; (2) the set $\mathfrak{A}_{qG}$ has nonempty $\tau$-interior; (3) the spectral radius $\rho(a)$ of any element $a \in \mathfrak{A}$ equals $\lim_{n\to\infty} \tau(a^n)^{1/n}$; (4) there is some constant $B$ such that $\rho(a) \leqq B\tau(a)$ for all $a \in \mathfrak{A}$. It is well known (and easy to show) that any maximal modular left ideal is closed in any $Q$-pseudo-norm.

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A $B^*$-algebra is a Banach *-algebra in which the complete norm is a $B^*$-pseudo-norm (and hence equals the Gelfand-Naimark pseudo-norm). For any Banach *-algebra $\mathfrak{A}$ let $\mathfrak{A}^-$ be the completion of $\mathfrak{A}/\mathfrak{A}_R$ in the norm induced by $\gamma$ where $\mathfrak{A}_R = \{a \in \mathfrak{A} : \gamma(a) = 0\}$. Let $\Phi : \mathfrak{A} \to \mathfrak{A}^-$ be the natural map. Then $\mathfrak{A}^-$ is a $B^*$-algebra and $(\Phi, \mathfrak{A}^-)$ is called the $B^*$-enveloping algebra of $\mathfrak{A}$. The states $\bar{\omega}$ of $\mathfrak{A}^-$ are in bijective correspondence to the states of $\mathfrak{A}$ under the map $\bar{\omega} \mapsto \bar{\omega} \circ \Phi$. Hence $\bar{\omega}$ is a pure state of $\mathfrak{A}^-$ iff $\bar{\omega} \circ \Phi$ is a pure state of $\mathfrak{A}$.

We need one less elementary fact. If $\mathfrak{A}$ is a $B^*$-algebra and $\mathfrak{L}$ is a closed left ideal in $\mathfrak{A}$ then $\mathfrak{L} = \bigcap \{\mathfrak{A}_\omega : \omega$ is a pure state of $\mathfrak{A}$ with $\mathfrak{L} \subseteq \mathfrak{A}_\omega\}$. For a proof see [**3**, 4.9.8], or [**1**, 5.4.5]. The same references may be consulted for further information on any of the concepts introduced above.

THEOREM. *Let $\mathfrak{A}$ be a Banach *-algebra. The following are equivalent*:

(a) *$\mathfrak{A}$ is hermitian.*

(b) *$\gamma$ is a $Q$-pseudo-norm.*

(c) *Every maximal modular left ideal in $\mathfrak{A}$ is $\gamma$-closed.*

(d) *Every maximal modular left ideal in $\mathfrak{A}$ is the left kernel of some pure state of $\mathfrak{A}$.*

PROOF. (a) $\Rightarrow$ (b): This follows immediately from the ingenious elementary proof by V. Ptak [**2**] that $\rho(a) \leqq \rho(a^*a)^{1/2} = \gamma(a)$ for all $a \in \mathfrak{A}$.

(b) $\Rightarrow$ (c): Well known and elementary.

(c) $\Rightarrow$ (d): Let $\mathfrak{L}$ be a maximal modular left ideal and hence $\gamma$-closed. Let $\mathfrak{L}^-$ be the closure of $\Phi(\mathfrak{L})$ in $\mathfrak{A}^-$. Then $\mathfrak{L}^-$ is a proper closed ideal and hence there is some pure state $\bar{\omega}$ on $\mathfrak{A}^-$ such that $\mathfrak{L}^- \subseteq \mathfrak{A}^-_{\bar{\omega}}$. Thus $\mathfrak{L} \subseteq \mathfrak{A}_\omega$ where $\omega = \bar{\omega} \circ \Phi$. However by the maximality of $\mathfrak{L}$, $\mathfrak{L} = \mathfrak{A}_\omega$. (If we knew only that $\bar{\omega}$ were a state on $\mathfrak{A}^-$ the usual argument would provide a pure state $\omega$ on $\mathfrak{A}$ with $\mathfrak{L} = \mathfrak{A}_\omega$.)

(d) $\Rightarrow$ (a): Suppose there is an element $b \in \mathfrak{A}$ and a nonzero positive number $t$ such that $t^{-1}b^*b$ is quasi-singular. Then either

$$\mathfrak{L} = \{ta + ab^*b : a \in \mathfrak{A}\} \quad \text{or} \quad \{ta + b^*ba : a \in \mathfrak{A}\}$$

is a proper ideal. However the second ideal is just $\mathfrak{L}^*$ so in either case $\mathfrak{L}$ is a proper modular left ideal with $-t^{-1}b^*b$ as a right relative unit. Thus there is a maximal modular left ideal including $\mathfrak{L}$ but not $b^*b$ and hence a pure state $\omega$ such that $\mathfrak{L} \subseteq \mathfrak{A}_\omega$ but $\omega((b^*b)^*b^*b) > 0$. However this is impossible since

$$\begin{aligned} 0 &< \omega((b^*b)^*b^*b) \\ &= (2t)^{-1}[\omega((tb + bb^*b)^*(tb + bb^*b)) \\ &\qquad - \omega((bb^*b)^*bb^*b) - t^2\omega(b^*b)] \leqq 0. \end{aligned}$$

REMARKS. (1) Notice that the proof (d) $\Rightarrow$ (a) establishes that $\mathfrak{A}$ is symmetric. Thus this is another way to derive the Shirali-Ford theorem from Ptak's proof of Raikov's inequality: $\rho(a) \leqq \rho(a^*a)^{1/2} = \gamma(a)$.

(2) Many properties of hermitian Banach *-algebras hold for all *-algebras in which the Gelfand-Naimark pseudo-norm is a $Q$-pseudo-norm. The proofs are frequently distinctly easier than proofs starting from the hypothesis of a hermitian Banach *-algebra. The theory of hermitian Banach *-algebras is developed from this viewpoint in [1].

(3) In particular if $\gamma$ is a $Q$-pseudo-norm on a *-algebra $\mathfrak{A}$ with $B^*$-enveloping algebra $(\Phi, \mathfrak{A}^-)$ then $\rho(a) = \lim_{n\to\infty}\gamma(a^n)^{1/n} = \rho(\Phi(a))$. Hence the convex hull of the spectrum of any element $a$ in $\mathfrak{A}$ agrees with the convex hull of the spectrum of $\Phi(a)$ in $\mathfrak{A}^-$. This provides a direct elementary proof (which avoids the result used to show (c) $\Rightarrow$ (d)) that condition (b) of the theorem implies symmetry and hence implies (a).

(4) Note that the proof (c) $\Rightarrow$ (d) actually establishes that every $\gamma$-closed left ideal $\mathfrak{L}$ in any *-algebra $\mathfrak{A}$ equals

$$\bigcap \{\mathfrak{A}_\omega : \omega \text{ a pure state of } \mathfrak{A} \text{ with } \mathfrak{L} \subseteq \mathfrak{A}_\omega\}.$$

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