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OPERATORS WITH DISCONNECTED SPECTRA ARE DENSE

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ABSTRACT. It is proven that the set of all (bounded linear) operators on a complex infinite dimensional Banach space having disconnected spectra is an open uniformly dense subset of the algebra of all operators.

In [3, Problem 8], P. R. Halmos asked whether the set of all reducible operators in a complex infinite dimensional separable Hilbert space $\mathscr{H}$ is uniformly dense in the algebra $\mathscr{L}(\mathscr{H})$ of all (bounded linear) operators on $\mathscr{H}$. In the present note we answer affirmatively a related question:

Is the set of all operators on a Banach space X having nontrivial complementary hyperinvariant subspaces dense in $\mathscr{L}(\mathscr{X})$? (Recall that a subspace $\mathscr{M}$ of $\mathscr{X}$ is *hyperinvariant* for $T \in \mathscr{L}(\mathscr{X})$ if $A\mathscr{M} \subset \mathscr{M}$, for all $A \in \mathscr{L}(\mathscr{X})$ commuting with T [1]. Here and in what follows, *subspace* means *closed linear manifold.*)

Moreover, we proved the following stronger (see [4]) result:

THEOREM. *Let $\mathscr{X}$ be a complex infinite dimensional Banach space and let $T \in \mathscr{L}(\mathscr{X})$. Then, given any $\varepsilon > 0$, there exists an $A \in \mathscr{L}(\mathscr{X})$ such that* (1) $\operatorname{rank}(A) = 1$; (2) $\|A\| < \varepsilon$, *and* (3) *the spectrum of $T + A$ is disconnected.*

PROOF. Let $\sigma(T)$ ($E(T)$, resp.) denote the spectrum (essential spectrum, resp.) of T.

Let λ_0 be any point of $E(T)$ such that $\mathbf{Re}\,\lambda_0 = \max\{\mathbf{Re}\,\lambda : \lambda \in E(T)\}$. Then, for every compact operator K, $\lambda_0 \in E(T + K) = E(T) \subset \sigma(T + K)$, and it follows from [4, *Theorem* 1] that, if there exists a $\lambda \in \sigma(T + K)$ such that $\mathbf{Re}\,\lambda > \mathbf{Re}\,\lambda_0$, then $\sigma(T + K)$ is disconnected, λ is an isolated point of $\sigma(T + K)$ such that $(T + K - \lambda)^n\mathscr{X}$ is closed for every $n \geqq 0$ and, if $\mathscr{M} = \bigcap_{n=1}^{\infty}(T + K - \lambda)^n\mathscr{X}$ and $\mathscr{N} = \operatorname{closure}\{\bigcup_{n=1}^{\infty} \ker(T + K - \lambda)^n\}$, then $\dim \mathscr{N} = \dim(\mathscr{X}/\mathscr{M}) < \infty$.

Therefore, to complete the proof, it suffices to find an A satisfying (1), (2) and such that $\lambda_0 + \gamma \in \sigma(T + A)$ for some γ, $0 < \gamma < \varepsilon/2$.

Since $\lambda_0 \in \operatorname{bdry} \sigma(T)$, there exists an $x \in \mathscr{X}$ such that $\|x\| = 1$ and $\|(T - \lambda_0)x\| < \varepsilon/2$ (see [2, *Chapter* 7]). By Hahn-Banach theorem, there

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exists a continuous linear functional f on $\mathscr{X}$ such that $f(x) = \|f\| = 1$. Define $P \in \mathscr{L}(\mathscr{X})$ by $Py = f(y)x$; then $\|P\| = 1$. If $y \in \mathscr{X}$ is a unit vector, then y can be written as $y = \alpha x + z$, where α is a complex number, $|\alpha| \leqq 1$, and $z \in \ker(f) = \ker(P)$.

For each γ, $0 < \gamma < \varepsilon/2$, define $T_\gamma \in \mathscr{L}(\mathscr{X})$ by $T_\gamma = T(I - P) + (\lambda_0 + \gamma)P$; then

$$(T_\gamma - T)y = [T(I - P) + (\lambda_0 + \gamma)P - T]y = (\lambda_0 + \gamma - T)Py$$
$$= \alpha(\lambda_0 + \gamma - T)x.$$

Hence $A_\gamma = T_\gamma - T = (\lambda_0 + \gamma - T)P$ has rank one and

$$\|A_\gamma\| = \sup\{\|(T_\gamma - T)y\| : \|y\| = 1\} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Clearly, $\lambda_0 + \gamma$ is an eigenvalue of T_γ and therefore $\lambda_0 + \gamma \in \sigma(T_\gamma)$. The proof is complete.

REMARK. If $\mathscr{M}$ and $\mathscr{N}$ are defined as above (for $\lambda = \lambda_0 + \gamma$ and $T_\gamma = T + A_\gamma$), then $\mathscr{M}, \mathscr{N}$ are hyperinvariant subspaces of T such that $\mathscr{X} = \mathscr{M} \oplus \mathscr{N}$; moreover, if γ is small enough, then $\dim \mathscr{N} = \dim \mathscr{X}/\mathscr{M} = 1$. With minor modifications of the same argument it is possible to show that, given $T \in \mathscr{L}(\mathscr{X})$ and $\varepsilon > 0$, there exists a compact operator K such that $\|K\| < \varepsilon$ and $\sigma(T + K)$ contains a sequence $\{\lambda_k : k = 1, 2, \ldots\}$ of isolated eigenvalues associated with hyperinvariant subspaces $\mathscr{N}_k, \mathscr{M}_k$ (defined as above) such that $\mathscr{X} = \mathscr{M}_k \oplus \mathscr{N}_k$ and $\dim \mathscr{N}_k = \dim(\mathscr{X}/\mathscr{M}_k) = 1$, for all k. From these results and [4, Theorem 3], we obtain the following:

COROLLARY. (1) *The set of all $T \in \mathscr{L}(\mathscr{X})$ such that $\sigma(T)$ is disconnected is a uniformly dense open subset of $\mathscr{L}(\mathscr{X})$.*

(2) *The set of all $T \in \mathscr{L}(\mathscr{X})$ such that, for each n ($n = 1, 2, 3, \ldots$), T has complementary hyperinvariant subspaces $\mathscr{N}_n$, $\mathscr{M}_n$ satisfying*

$$\dim \mathscr{N}_n = \dim(\mathscr{X}/\mathscr{M}_n) = n,$$

is dense in $\mathscr{L}(\mathscr{X})$.

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