OPERATORS WITH DISCONNECTED SPECTRA ARE DENSE

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Communicated by Fred Gehring, December 13, 1971

ABSTRACT. It is proven that the set of all (bounded linear) operators on a complex infinite dimensional Banach space having disconnected spectra is an open uniformly dense subset of the algebra of all operators.

In [3, Problem 8], P. R. Halmos asked whether the set of all reducible operators in a complex infinite dimensional separable Hilbert space $H$ is uniformly dense in the algebra $\mathcal{L}(H)$ of all (bounded linear) operators on $H$. In the present note we answer affirmatively a related question:

Is the set of all operators on a Banach space $X$ having nontrivial complementary hyperinvariant subspaces dense in $\mathcal{L}(X)$? (Recall that a subspace $M$ of $X$ is hyperinvariant for $T \in \mathcal{L}(X)$ if $AM \subseteq M$, for all $A \in \mathcal{L}(X)$ commuting with $T$ [1]. Here and in what follows, subspace means closed linear manifold.)

Moreover, we proved the following stronger (see [4]) result:

THEOREM. Let $X$ be a complex infinite dimensional Banach space and let $T \in \mathcal{L}(X)$. Then, given any $\varepsilon > 0$, there exists an $A \in \mathcal{L}(X)$ such that (1) rank $(A) = 1$; (2) $\|A\| < \varepsilon$, and (3) the spectrum of $T + A$ is disconnected.

PROOF. Let $\sigma(T)$ ($E(T)$, resp.) denote the spectrum (essential spectrum, resp.) of $T$.

Let $\lambda_0$ be any point of $E(T)$ such that $\text{Re} \lambda_0 = \max \{\text{Re} \lambda; \lambda \in E(T)\}$. Then, for every compact operator $K$, $\lambda_0 \in E(T + K) = E(T) \subset \sigma(T + K)$, and it follows from [4, Theorem 1] that, if there exists a $\lambda \in \sigma(T + K)$ such that $\text{Re} \lambda > \text{Re} \lambda_0$, then $\sigma(T + K)$ is disconnected, $\lambda$ is an isolated point of $\sigma(T + K)$ such that $(T + K - \lambda)^nX$ is closed for every $n \geq 0$ and, if $M = \bigcap_{n=1}^{\infty} (T + K - \lambda)^nX$ and $N = \text{closure} \{\bigcup_{n=1}^{\infty} \text{ker}(T + K - \lambda)^n\}$, then $\dim N = \dim (X/M) < \infty$.

Therefore, to complete the proof, it suffices to find an $A$ satisfying (1), (2) and such that $\lambda_0 + \gamma \in \sigma(T + A)$ for some $\gamma$, $0 < \gamma < \varepsilon/2$.

Since $\lambda_0 \in \text{bdry} \sigma(T)$, there exists an $x \in X$ such that $\|x\| = 1$ and $\|(T - \lambda_0)x\| < \varepsilon/2$ (see [2, Chapter 7]). By Hahn-Banach theorem, there...
exists a continuous linear functional \( f \) on \( \mathcal{X} \) such that \( f(x) = \| f \| = 1 \). Define \( P \in \mathcal{L}(\mathcal{X}) \) by \( Py = f(y)x \); then \( \| P \| = 1 \). If \( y \in \mathcal{X} \) is a unit vector, then \( y \) can be written as \( y = \alpha x + z \), where \( \alpha \) is a complex number, \( |\alpha| \leq 1 \), and \( z \in \ker(f) = \ker(P) \).

For each \( \gamma, \ 0 < \gamma < \varepsilon/2 \), define \( T_\gamma \in \mathcal{L}(\mathcal{X}) \) by \( T_\gamma = T(I - P) + (\lambda_0 + \gamma)P \); then

\[
(T_\gamma - T)y = [T(I - P) + (\lambda_0 + \gamma)P - T]y = (\lambda_0 + \gamma - T)Py
= \alpha(\lambda_0 + \gamma - T)x.
\]

Hence \( A_\gamma = T_\gamma - T = (\lambda_0 + \gamma - T)P \) has rank one and

\[
\| A_\gamma \| = \sup\{\| (T_\gamma - T)y \| : \| y \| = 1 \} < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Clearly, \( \lambda_0 + \gamma \) is an eigenvalue of \( T_\gamma \) and therefore \( \lambda_0 + \gamma \in \sigma(T_\gamma) \).

The proof is complete.

REMARK. If \( \mathcal{M} \) and \( \mathcal{N} \) are defined as above (for \( \lambda = \lambda_0 + \gamma \) and \( T_\gamma = T + A_\gamma \)), then \( \mathcal{M}, \mathcal{N} \) are hyperinvariant subspaces of \( T \) such that \( \mathcal{X} = \mathcal{M} \oplus \mathcal{N} \); moreover, if \( \gamma \) is small enough, then \( \dim \mathcal{N} = \dim \mathcal{X}/\mathcal{M} = 1 \). With minor modifications of the same argument it is possible to show that, given \( T \in \mathcal{L}(\mathcal{X}) \) and \( \varepsilon > 0 \), there exists a compact operator \( K \) such that \( \| K \| < \varepsilon \) and \( \sigma(T + K) \) contains a sequence \( \{ \lambda_k : k = 1, 2, \ldots \} \) of isolated eigenvalues associated with hyperinvariant subspaces \( \mathcal{N}_k, \mathcal{M}_k \) (defined as above) such that \( \mathcal{X} = \mathcal{M}_k \oplus \mathcal{N}_k \) and \( \dim \mathcal{N}_k = \dim(\mathcal{X}/\mathcal{M}_k) = 1 \), for all \( k \). From these results and [4, Theorem 3], we obtain the following:

**Corollary.**

(1) The set of all \( T \in \mathcal{L}(\mathcal{X}) \) such that \( \sigma(T) \) is disconnected is a uniformly dense open subset of \( \mathcal{L}(\mathcal{X}) \).

(2) The set of all \( T \in \mathcal{L}(\mathcal{X}) \) such that, for each \( n \) (\( n = 1, 2, 3, \ldots \)), \( T \) has complementary hyperinvariant subspaces \( \mathcal{N}_n, \mathcal{M}_n \) satisfying

\[
\dim \mathcal{N}_n = \dim(\mathcal{X}/\mathcal{M}_n) = n,
\]

is dense in \( \mathcal{L}(\mathcal{X}) \).

**References**


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