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HOMOTOPY GROUPS OF FINITE *H*-SPACES

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In this announcement we present results about the homotopy groups of H-spaces having the homotopy type of finite CW-complexes. We call such spaces *finite H-spaces.* We always assume our spaces are connected. In the sequel we always use X to denote a finite H-space. In some statements we refer to a direct sum of cyclic groups. We do not rule out the case that the sum is zero.

Let $\tilde{X}$ be the fibre of the canonical map

$$X \to K(\Pi_1(X), 1).$$

It is well known that this "universal covering space" $\tilde{X}$ is a finite H-space.

THEOREM 1. *$\Pi_4(X)$ is a direct sum of groups of order* 2, $\dim \Pi_4(X)' = \dim \ker \mathrm{Sq}^2 : H^3(\tilde{X} : Z_2) \to H^5(\tilde{X} : Z_2)$.

PROOF. Since $\tilde{X}$ is a finite H-space, it suffices to work with simply connected X. We use the exact sequence of J. H. C. Whitehead,

$$\to H_{n+1}(X:Z) \xrightarrow{\nu_n} \Gamma_n(X) \xrightarrow{\lambda_n} \Pi_n(X) \xrightarrow{h_n} H_n(X:Z) \to .$$

Results of Browder [3] and Hilton [7] give $\Gamma_4(X) \cong H_3(X:Z_2)$. Browder's Theorem 6.1 of [3] yields

LEMMA 2. *Let X be simply connected, then $H_4(X:Z) = 0$.*

From [7] we obtain ν_4 as the composite

$$H_5(X:Z) \xrightarrow{r} H_5(X:Z_2) \xrightarrow{\mathrm{Sq}^2_*} H_3(X:Z_2)$$

where r is reduction mod 2. The theorem follows.

We remark that if X is simply connected and $H_*(\Omega X:Z)$ torsion free, then Theorem 1 is contained in Bott-Samelson [2].

For the remainder of this paper we assume that *X is simply connected and $H_*(\Omega X:Z)$ is torsion free.* We identify $\Gamma_4(X)$, $H_3(X:Z_2)$ and $\Pi_3(X) \otimes Z_2$, and continue to use ν_4. For $k \geqq 3$, $\eta_k : S^{k+1} \to S^k$ is the essential map.

THEOREM 3. *The following sequence is exact,*

$$0 \to \Pi_4(X) \xrightarrow{\eta_4^*} \Pi_5(X) \xrightarrow{h_5} H_5(X:Z) \xrightarrow{\nu_4} \Pi_3(X) \otimes Z_2 \xrightarrow{\eta_3^*} \Pi_4(X) \to 0,$$

with $\ker h_5 = \operatorname{tors} \Pi_5(X)$, *the torsion subgroup of* $\Pi_5(X)$.

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OUTLINE OF PROOF. In the appropriate segment of the Whitehead sequence, use [7] to show $\lambda_5\Gamma_5(X) \cong \Pi_4(X)$. From the Cartan-Serre Theorem [9] we have $\ker h_5 \subset \text{tors}\,\Pi_5$. To prove the opposite inclusion we first use a theorem of Clark [6] which yields the fact that the p-torsion of $H_*(X:Z)$ is of order at most p. Applying a theorem of Browder [4] gives $H_5(X:Z) = F \oplus T$ where F is free and T is a direct sum of cyclic groups of order 2. We then use arguments involving the Serre spectral sequence to show that if $h_5(\text{tors}\,\Pi_5(X)) \neq 0$ then $H_*(\Omega X:Z)$ has torsion. The remaining details are straightforward.

Further use of the Whitehead sequence and [7] yields

THEOREM 4. *Let p be a prime. If $p \geqq 5$, then $\Pi_6(X)$ is p-torsion free. The 3-torsion is of order at most 3 and the 2-torsion of order at most 4.*

More detailed information can be obtained by means of the Massey-Peterson spectral sequence [8] and its extensions to odd primes [5]. The hypotheses for the use of the spectral sequence include $H^*(X:Z_p) = \bigcup(M)$ as algebras over the Steenrod algebra. Many H-spaces satisfy this but I know of no general result for finite H-spaces. However, if one can prove that $H^*(X:Z_p)$ satisfies this condition through a range of dimensions, then the spectral sequence can be used to calculate homotopy groups in a slightly smaller range. Via this technique, we obtain the following results:

THEOREM 5. *Let p be a prime. Then $\Pi_n(X)$ is p-torsion free for $n < 2p$ and the p-torsion of $\Pi_{2p}(X)$ is of order at most p. Furthermore, for odd primes,* $\dim \Pi_{2p}(X) \otimes Z_p = \dim \ker P^1 : H^3(X:Z_p) \to H^{2p+1}(X:Z_p)$.

Our remaining results require a hypothesis in addition to those already carried. Equivalent forms are given in the next statement.

PROPOSITION 6. *The following statements are equivalent*:
(a) $H^5(X:Z_2) = \text{Sq}^2 H^3(X:Z_2)$;
(b) $\text{im}\, h_5 = 2H_5(X:Z)$;
(c) *the 5-skeleton X^5 is a bouquet of types S^3 and $S^3 \cup_{\eta_3} e^5$*;
(d) $\dim \Pi_4(X) = \dim H_3(X:Z_2) - \dim H_5(X:Z_2)$.

We conjecture that these statements are true in general.

THEOREM 7. *Assume the statements of Proposition 6 are true. Then*

$$\dim \Pi_6 \otimes Z_2 \leqq \dim[(\ker \text{Sq}^3 \cap \ker \text{Sq}^4\,\text{Sq}^2)H^3(X:Z_2)]$$

the torsion subgroup of $\Pi_7(X)$ is a direct sum of cyclic groups of order 2.

The statement for Π_6 means "the dimension of the intersection of the kernals of the listed cohomology operations when applied to $H^3(X:Z_2)$." The proofs of Theorem 5 and the part about Π_7 essentially involve only the calculation of E_2 of the spectral sequence. The part about Π_6 involves a differential.

In summary, we list in tabular form the structure of the first seven homotopy groups. The table is for H-spaces X such that $H^*(\Omega\tilde{X}:Z_2)$ is torsion free and $\tilde{X}$ satisfies Proposition 6. We use F to mean a free group and T_n a direct sum of cyclic groups of order n. Assuming Proposition 6 allows us to improve Theorems 1 and 3.

n	Π_n	*Remark*
1	any finitely generated abelian group	[1]
2	0	[3]
3	F	
4	T_2	
5	$F \oplus T_2$	
6	$T_2 \oplus T_3 \oplus T_4$	
7	$F \oplus T_2$	$\dim T_2 \leqq \operatorname{rank} \Pi_3$

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