BASIC ELEMENTS: THEOREMS FROM ALGEBRAIC
K-THEORY

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1. **Introduction.** Several of the fundamental theorems about algebraic K_0 and K_1 are concerned with finding unimodular elements, that is, elements of a projective module which generate a free summand. In this announcement we use the notion of a *basic element* (in the terminology of Swan [5]) to extend these theorems to the context of finitely generated modules which may not be projective. Our techniques allow a simplification and strengthening of the existing results even in the projective case. Our Theorem A includes extensions of Serre's theorem on free summands of "large" projective modules, Bass' cancellation and stable range theorems, and the theorem of Forster and Swan on the number of generators of a module. Theorem B contains a further extension of the Forster-Swan theorem and the stable range theorem.

The proofs of our theorems, which will appear elsewhere, involve a mixture of the methods of Swan [5] and Bass [1, Theorem 8.2]. By making heavy use of the generic points provided by the j-spectrum, Swan's method enables one to simplify Bass' proofs and obtain stronger results.

2. **Preliminaries.** Throughout this paper, rings have units and modules are finitely generated. R will always denote a commutative ring with noetherian maximal spectrum, and A will denote an R algebra which is finitely generated as an R-module.

The theorems of Bass and Serre that we have mentioned were proved using only the space of maximal ideals of R. To extend these results, it is necessary to work with all the j-primes of R; that is, those primes of R which are intersections of maximal ideals. (See [5] for a detailed exposition of the relation with the maximal spectrum.) Following Swan [5] we will say that the j-dimension of a j-prime is the length of the longest chain of j-primes containing it. Other j-concepts are defined similarly. The fundamental notions of this paper are the following:

DEFINITION. Let M be an A-module, $m \in M$, and x a prime ideal of R. m *is basic in* M *at* x if m can appear as part of a minimal system of generators for the A_x-module M_x (in particular $M_x \neq 0$).

m *is j-basic in* M if m is basic in M at every j-prime of R.

If $M' \subset M$ is a submodule, then M' *is n-fold basic in* M *at* x if the minimal

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number of generators of $(M/M')_x$ is at least n smaller than the minimal number of generators of M_x (as A_x-modules).

3. Results.

THEOREM A. (i) *Let $d = j$-dim R. Suppose that M is an A-module which cannot be generated by fewer than $d + 1$ elements locally at any j-prime of R. Then M contains a j-basic element.*

(ii) *More generally, if $M' \subseteq M$ are A-modules such that at each j-prime x of E, M' is $[(j$-dim $x) + 1]$-fold basic in M, then M' contains a j-basic element of M. If $m_1, \ldots, m_k \in M'$ generate M', and if $a \in A$ is such that $(a, m_1) \in A \oplus M$ is j-basic, then the j-basic element of M in M' may be chosen to have the form*

$$m_1 + aa_2 m_2 + \cdots + aa_k m_k$$

for some $a_i \in A$.

REMARKS. (1) The first statement of (ii) follows from the second by letting $m_1, \ldots, m_k$ be a set of generators for M and letting $M = M'$, and $a = 1$.

(2) The basicness conditions of part (i) only need to be checked at the finitely many minimal j-primes of R.

(3) If one erases the letter j each time it occurs in the theorem, one obtains a different true statement.

(4) The first statement of the theorem was suggested to us by M. Artin who kindly showed us an unpublished manuscript containing a different proof of the commutative case.

COROLLARY 1 (*Serre's theorem* [4, *Theorem* 1]). *Let j-dim $R = d$. If P is a projective A-module whose rank at every localization is at least $d + 1$, then P has a free direct summand. If $m_1, \ldots, m_n$ generate P, then the generator of the free direct summand may be chosen to be of the form*

$$m = m_1 + a_2 m_2 + \cdots + a_n m_n$$

with $a_i \in A$.

What Serre actually proved was the first statement of the corollary, but only for commutative rings. Bass extended this to the noncommutative case in [1]. The second statement was proved by Murthy [3, Lemma 3] in the case $A = R$, $d = 1$.

SKETCH OF PROOF. Choose a projective A-module Q so that $P \oplus Q$ is free, and apply part (ii) of Theorem A to the case $a = 1$, $M = P \oplus Q$, $M' = P$. Note that the hypothesis of Corollary 1 insures that P is $[d + 1]$-fold basic in $P \oplus Q$. A basic element of a free module always generates a free direct summand.

The next corollary is the heart of Bass' cancellation theorem [1, Theorem 9.3]. The cancellation theorem itself follows immediately from this statement.

COROLLARY 2 *Let j-dim $R = d$, let N be any module, and Q be a projective A-module whose rank at each localization is at least $d + 1$. Let (p, q, n) $\in P \oplus Q \oplus N$ generate a free direct summand. Then there is a map $f : P \to Q$ such that $(f(p) + q, n) \in Q \oplus N$ generates a free direct summand of $Q \oplus N$.*

The proof uses Theorem A, part (ii) and the technique of "order ideals" as in Swan [5, Chapter 12].

COROLLARY 3 (*Forster-Swan Theorem on the number of generators of a module*). *Let N be a finitely generated A-module and suppose that*

$$k = \max_{x \ a \ j\text{-}prime \ of \ R} \{(j\text{-dim } x)$$

$$+ \ (\text{the minimal number of generators of } M_x \text{ over } A_x)\}.$$

Then N can be generated by k elements.

PROOF. By passing to $A/\text{ann}(N)$ we can assume N is faithful. Let $0 \to M' \to M \to N \to 0$ be exact with M free. If rank $M > k$, then Theorem A, part (ii) shows that M' contains a j-basic element $m \in M$. Since the hypotheses of the corollary force $k > (j\text{-dim } R) + 1$, Bass' cancellation theorem shows that M/Am is a free module of rank (rank M) $- 1$ which maps onto N. Induction finishes the proof.

COROLLARY 4 (*Bass' stable range theorem* [6, *Theorem* 12.3]). *Let $n_1, \ldots, n_k \in A$ be such that the right ideal generated by $(n_1, \ldots, n_k)$ is A and such that $k > (j\text{-dim } R) + 1$. Then there exist $a_1, \ldots, a_{k-1} \in A$ such that $n_1 + a_1 n_k, \ldots, n_{k-1} + a_{k-1} n_k$ generate A as a right ideal.*

PROOF. Apply Theorem A, part (ii) to the setup $M = A^{k-1}$, $a = n_k$, $m_1 = (n_1, \ldots, n_{k-1}) \in A^{k-1}$, and $m_2, \ldots, m_{k+1}$ basis elements of A^k, noting that an element of a free module is j-basic if and only if its components generate the unit ideal.

The last two corollaries admit a common generalization in a direction different from that of Theorem A:

THEOREM B. *Let M be an A-module, and let $m_1, \ldots, m_k$ generate M. Suppose that for every j-prime x of R such that $M_x \neq 0$, we have*

$$k > (j\text{-dim } x) + (\text{the minimal number of generators of } M_x \text{ over } A_x).$$

Then there exist elements $a_1, \ldots, a_{k-1} \in A$ such that $m_1 + a_1 m_k, \ldots, m_{k-1} + a_{k-1} m_k$ generate M.

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