

ON THE REALIZABILITY OF MODULES OVER THE STEENROD ALGEBRA

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1. Introduction. Let A denote the mod 2 Steenrod algebra. There is good reason to believe that knowledge both of the structure of the category of A -modules and of the subcategory of those A -modules which are the cohomology of spectra (i.e. are realizable), will give us great insight into the structure of the stable homotopy category of spaces. So far the structure and realizability problems have received little attention of a general nature, work having mostly centered around the specific insights needed to attack specific problems. For example, with respect to the realizability problem, research to date gives us little more than the realizability of specific A -modules (e.g. [2], [3] and [4]).

This paper describes a step in the attack of these problems from a more general point of view.

I begin by recalling work by J. F. Adams and myself [1] concerning the structure problem. Using this work I describe a family of natural constructions analogous to the killing of homotopy groups. These constructions can be performed both on A -modules and on spectra, and under mod 2 cohomology they correspond. Therefore if an A -module is realizable then so are the modules obtained from it by the constructions.

A more detailed exposition of this work and its ramifications is in preparation.

2. Modules over the Steenrod algebra. The structure of the mod 2 Steenrod algebra, A , is well known and we are in particular interested in a family of elements best describable in terms of the Milnor basis [5]: Let P_i^s be the dual of $\xi_i^{2^s}$. Then an easy computation using the product formula gives us that $(P_i^s)^2 = 0$ if $s < t$. This allows the definition:

DEFINITION 2.1. Let M be a left A -module; if $s < t$ then define $H(M, P_i^s) = \ker P_i^s | M / P_i^s M$, the P_i^s -homology group of M . This is clearly a functorial invariant of M . It is also graded by the grading on M . In what follows we will only consider P_i^s with $s < t$.

These are strong invariants as evidenced by the following:

THEOREM 2.2 [1]. *If $f: M \rightarrow N$ is a map of bounded below (b.b.) A -modules and $H(f, P_i^s)$ is an isomorphism for all $s < t$ then M and N are stably iso-*

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morphic, i.e. there are free A -modules F and G such that $M \oplus F$ and $N \oplus G$ are isomorphic.

They are also very well behaved and in particular we have the following results needed in §3.

THEOREM 2.3. *Let B be a sub-Hopf algebra of A and let $M = A \otimes_B N$ for a B -module N . If $P_t^s \notin B$ then $H(M, P_t^s) = 0$ and if $P_t^s \in B$ there is a spectral sequence with $E^1 = (A \otimes_B Z_2) \otimes H(N, P_t^s)$ and $E^\infty = H(M, P_t^s)$.*

THEOREM 2.4. *Given $f \otimes g: M_1 \otimes M_2 \rightarrow N_1 \otimes N_2$; if $H(f, P_t^s)$ and $H(g, P_t^s)$ are isomorphisms then $H(f \otimes g, P_t^s)$ is an isomorphism.*

3. Killing homology groups. Let X^* denote $H^*(X; Z_2)$ —this conforms pleasantly with the usual notation f^* for maps. The homology groups of §2 give us invariants of spectra if we consider $H(X^*, P_t^s)$. These invariants have the same strength in the topological setting as they did in the algebraic:

THEOREM 3.1. *Let $f: X \rightarrow Y$ be a map of b.b. spectra such that $H(f^*, P_t^s)$ is an isomorphism for all P_t^s , then X and Y are stably equivalent, i.e. homotopy equivalent up to $K(Z_2)$ factors.*

The constructions alluded to in §1 play the same role for the homology groups as the constructions of “killing homotopy groups” play for homotopy groups. The implied ordering is *not* the grading of an individual homology group but the ordering by degree of the P_t^s s. Note that $P_{t_1}^{s_1} \neq P_{t_2}^{s_2}$ implies $\text{deg } P_{t_1}^{s_1} \neq \text{deg } P_{t_2}^{s_2}$ so this ordering is linear.

DEFINITION 3.2. Let X be a spectrum, then a spectrum Y is of type $X \langle 0, P_t^s \rangle$ (resp. type $X \langle P_t^s, \infty \rangle$) if there is a map $f: X \rightarrow Y$ (resp. $f: Y \rightarrow X$) such that $H(f^*, P_{t_1}^{s_1})$ is an isomorphism if $\text{deg } P_{t_1}^{s_1} \leq \text{deg } P_t^s$ (resp. $\text{deg } P_{t_1}^{s_1} \geq \text{deg } P_t^s$) and $H(Y^*, P_{t_1}^{s_1}) = 0$ if $\text{deg } P_{t_1}^{s_1} > \text{deg } P_t^s$ (resp. $\text{deg } P_{t_1}^{s_1} < \text{deg } P_t^s$).

The main result is the existence, uniqueness and naturality of these constructions.

THEOREM 3.3. (a) *For any (b.b.) spectrum X and any P_t^s there are (b.b.) spectra of type $X \langle 0, P_t^s \rangle$ and of type $X \langle P_t^s, \infty \rangle$.*

(b) *If Y_1 and Y_2 are b.b. spectra of type $X \langle 0, P_t^s \rangle$ or $X \langle P_t^s, \infty \rangle$ then Y_1 and Y_2 are stably equivalent.*

(c) *If $f: X_1 \rightarrow X_2$ is a map of spectra then there are spectra Y_1 and Y_2 of types $X_1 \langle 0, P_t^s \rangle$ and $X_2 \langle 0, P_t^s \rangle$, and a map $g: Y_1 \rightarrow Y_2$ such that*

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

homotopy commutes. Similarly for type $X \langle P_t^s, \infty \rangle$.

The same types of constructions can be performed for A -modules in the same way.

THEOREM 3.4. *If M is a realizable A -module then so are modules of types $M\langle 0, P_t^s \rangle$ and $M\langle P_t^s, \infty \rangle$.*

Let me outline the proof of the existence of a spectrum of type $X\langle P_t^s + 1, \infty \rangle$ ($P_t^s + 1$ denotes the successor of P_t^s in the ordering by degree). Let A_n be the sub-Hopf algebra of A generated by the first $n + 1$ indecomposables: $P_1^0, P_1^1, \dots, P_1^n$. Then $P_t^s \in A_n$ if and only if $s + t \leq n + 1$ and from this it follows that each A_n contains an initial segment of the P_t^s 's ordered by degree—specifically $\{P_t^s | \deg P_t^s \leq \deg P_{n+1}^0\}$. Our first step is to prove the algebraic version of 3.3(a) for A_n -modules.

THEOREM 3.5. *Given an A_n -module M and $P_t^s \in A_n$ there is an A_n -module and monomorphism $f: M \rightarrow N$ such that*

- (a) $H(f, P_{t_1}^{s_1})$ is an isomorphism if $\deg P_{t_1}^{s_1} > \deg P_t^s$,
- (b) $H(N, P_{t_1}^{s_1}) = 0$ if $\deg P_{t_1}^{s_1} \leq \deg P_t^s$.

Further if M is bounded below then so is N .

We construct $X\langle P_t^s + 1, \infty \rangle$ as an inverse limit using

LEMMA 3.6. *Let X be a b.b. spectrum then there is a spectrum and map $f: Y \rightarrow X$ such that*

- (a) $H(f^*, P_{t_1}^{s_1})$ is an isomorphism if $\deg P_{t_1}^{s_1} > \deg P_t^s$,
- (b) if $\deg P_{t_1}^{s_1} \leq \deg P_t^s$ and $H_i(X^*, P_{t_1}^{s_1}) = 0$ for $i < w$ then $H_i(Y^*, P_{t_1}^{s_1}) = 0$ for $i \leq w$.

In addition we can get f to be a homotopy equivalence through $\deg(w + u)$ with u independent of w .

OUTLINE OF PROOF. If $P_t^s \in A_n$ then regarding X^* as an A_n -module we have by 3.5 the exact sequence of A_n -modules $0 \rightarrow X^* \xrightarrow{f} N \rightarrow L \rightarrow 0$ such that, for $P_{t_1}^{s_1} \in A_n$,

- (a) $H(f, P_{t_1}^{s_1})$ is an isomorphism if $\deg P_{t_1}^{s_1} > \deg P_t^s$ and
- (b) $H(N, P_{t_1}^{s_1}) = 0$ if $\deg P_{t_1}^{s_1} < \deg P_t^s$.

Let $F_1 \xrightarrow{d} F_2 \rightarrow L \rightarrow 0$ be an exact sequence of A_n -modules with F_1 and F_2 free. There are also maps $g: F_1 \rightarrow X^*$ and $h: F_2 \rightarrow N$ giving a commutative diagram. Then

$$A \otimes_{A_n} F_1 \xrightarrow{1 \otimes d} A \otimes_{A_n} F_2 \rightarrow A \otimes_{A_n} L \rightarrow 0$$

is an exact sequence of A -modules and the map g induces an A -map $g_1: A \otimes_{A_n} F_1 \rightarrow X^*$.

We now apply a well-known theorem and get $A \otimes_{A_n} F_1 = K_1^*$, $A \otimes_{A_n} F_2 = K_2^*$, and maps $K_2 \rightarrow K_1$ and $X \rightarrow K_1$ inducing $1 \otimes d$ and g_1 in cohomology. Then Y is defined as the pullback in

$$\begin{array}{ccc} Y & \rightarrow & K_2 \\ \downarrow & & \downarrow \\ X & \rightarrow & K_1. \end{array}$$

If $Z \rightarrow K_2 \rightarrow K_1$ is exact then so is $Z \rightarrow Y \rightarrow X$ and in addition it is possible to show that in this case $0 \rightarrow X^* \rightarrow Y^* \rightarrow Z^* \rightarrow 0$ is exact. So the lemma will be proved if we can show that $H(Z^*, P_{i_1}^{s_1}) = 0$ for $\deg P_{i_1}^{s_1} > \deg P_i^s$ and that, for $i \leq t$, $\delta: H_i(Z^*, P_{i_1}^{s_1}) \rightarrow H_{i+p}(X^*, P_{i_1}^{s_1})$ ($p = \deg P_{i_1}^{s_1}$) is an isomorphism if $\deg P_{i_1}^{s_1} \leq \deg P_i^s$. Let me indicate the argument used to prove the first of these assertions. Since $Z \rightarrow K_2 \xrightarrow{\cong} K_1$ is exact, Z^* is an extension of $\text{coker } h^* = A \otimes_{A_n} L$ by $\ker h^*$. Recall that $0 \rightarrow X^* \rightarrow N \rightarrow L \rightarrow 0$ is exact, so using Theorem 2.3 we can show that $H(A \otimes_{A_n} L, P_{i_1}^{s_1}) = 0$ if $\deg P_{i_1}^{s_1} > \deg P_i^s$. This implies the same for $\ker h^*$ since $0 \rightarrow \ker h^* \rightarrow K_2 \rightarrow \text{im } h^* \rightarrow 0$ and $0 \rightarrow \text{im } h^* \rightarrow K_1 \rightarrow \text{coker } h^* \rightarrow 0$ are both exact. And therefore the exactness of $0 \rightarrow \text{coker } h^* \rightarrow Z^* \rightarrow \ker h^* \rightarrow 0$ gives us that $H(Z^*, P_{i_1}^{s_1}) = 0$ if $\deg P_{i_1}^{s_1} > \deg P_i^s$. A more complicated argument is needed to prove the other assertion.

4. Postscript. As with the analogous work with homotopy groups, the results described in §3 immediately imply a decomposition of arbitrary spectra into a tower of fibrations with fibre at each stage a spectrum whose cohomology has only one nonvanishing homology group. In a sequel to this paper I will describe my work with such spectra. Suffice it to say that they enjoy some very interesting properties.

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