BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

A NEW EXACT SEQUENCE FOR K_2 AND SOME CONSEQUENCES FOR RINGS OF INTEGERS

BY R. KEITH DENNIS[1] AND MICHAEL R. STEIN[2]

Communicated by Hyman Bass, November 29, 1971

Suppose R is a Dedekind domain with field of fractions F and at most countably many maximal ideals P. Using methods from the theory of algebraic groups, Bass and Tate [B-T] have proved the exactness of the sequence

$$K_2(R) \to K_2(F) \xrightarrow{t} \coprod_P K_1(R/P) \to K_1(R) \to K_1(F) \to \cdots$$

where t is induced by the tame symbols on R. They have also asked whether this sequence remains exact with "$0 \to$" inserted on the left when R is a ring of algebraic integers. In this note we announce an affirmative response when R is a discrete valuation ring, and a proof that the resulting sequence is split exact under certain additional hypotheses on R. In addition, we derive consequences of these results for a ring, $\mathfrak{O}$, of integers in a number field. Among these are

(1) a complete determination of the groups $K_2(\mathfrak{O}/\mathfrak{a})$ for any ideal $\mathfrak{a}$ of $\mathfrak{O}$; and

(2) examples of rings of integers $\mathfrak{O}$ for which $K_2(\mathfrak{O})$ is not generated by symbols and $K_2(2, \mathfrak{O}) \to K_2(3, \mathfrak{O})$ is not surjective. Detailed proofs will appear elsewhere.

1. **The exact sequence.** Let A be a discrete valuation ring with field of fractions K and residue field k. Define the tame symbol [Mi, Lemma 11.4] $t: K_2(K) \to K_1(k) \approx k^*$ by $t(\{u\pi^i, v\pi^j\}) = (-1)^{ij}\bar{u}^j\bar{v}^{-i}$, $u, v \in A^*$, where π generates the maximal ideal of A.

THEOREM 1. *The sequence*

$$0 \to K_2(A) \to K_2(K) \xrightarrow{t} K_1(k) \to 0$$

is exact. Moreover, if A is complete and k is perfect, this sequence is split exact.

The methods used in this proof are elementary in the sense that they

---

AMS 1970 *subject classifications.* Primary 18F25, 13D15, 20G25, 20G30; Secondary 12B25, 13J10.

Key words and phrases. K_2, discrete valuation ring, universal for GE_n, stability theorems for K_2, algebraic integers, roots of unity, tame symbol, Steinberg symbol.

[1] Part of this research was done while the first author was a visiting member supported by the Institute for Advanced Study.

[2] Supported by NSF-GP-28915. The second author wishes to thank the Institute for Advanced Study for their hospitality during part of this research.

use no machinery from the theory of algebraic groups. Split exactness is proved by explicit construction of a splitting homomorphism $\rho: K_2(K) \to K_2(A)$, using several new identities satisfied by Steinberg symbols in $K_2(A)$. The proof that $K_2(A) \to K_2(K)$ is injective in the general case uses these new identities and the Reidemeister-Schreier method for obtaining presentations of subgroups [M-K-S, §2.3].

The proof of Theorem 1 depends, in the language of Chevalley groups, only on the presence of a root system of type A_2. Thus Theorem 1 holds for the groups $L(\Phi, A)$ defined in [St1, (3.10)] whenever Φ is nonsymplectic. In particular, Theorem 1 holds for the groups $K_2(n, A) = L(A_{n-1}, A)$, $n \geqq 3$, of [D]. By keeping track of which new identities are used in the proof of Theorem 1, we obtain

THEOREM 2. *If A is a discrete valuation ring and $n \geqq 3$, $K_2(n, A)$ has a presentation with generators $\{u, v\}$, $u, v \in A^*$, subject to the Steinberg-Matsumoto relations* [Ma, *Lemme* 5.6] *and three additional relations, as follows*:

(1) $\{u, vw\} = \{u, v\}\{u, w\}$, $w \in A^*$.
(2) $\{u, v\} = \{v, u\}^{-1}$.
(3) $\{u, -u\} = 1$.
(4) $\{u, 1 - u\} = 1$, *if* $1 - u \in A^*$.

$$\text{(5)}\quad \begin{aligned} &\left\{u_1, 1 + qu_1\right\}\left\{\frac{u_2}{1 + qu_1}, \frac{1 + q(u_1 + u_2)}{1 + qu_1}\right\} \\ &\qquad = \left\{v_1, 1 + qv_1\right\}\left\{\frac{v_2}{1 + qv_1}, \frac{1 + q(v_1 + v_2)}{1 + qv_1}\right\} \end{aligned}$$

for $q \in \operatorname{rad} A, u_1, u_2, v_1, v_2 \in A^*$ *such that* $u_1 + u_2 = v_1 + v_2 \notin A^*$.

$$\text{(6)}\quad \{v, 1 - pqv\} = \left\{-\frac{1 - qv}{1 - p}, \frac{1 - pqv}{1 - p}\right\}\left\{-\frac{1 - pv}{1 - q}, \frac{1 - pqv}{1 - q}\right\}$$

for $p, q \in \operatorname{rad} A$.

$$\text{(7)}\quad \left\{-\frac{1-qr}{1-p}, \frac{1-pqr}{1-p}\right\}\left\{-\frac{1-pr}{1-q}, \frac{1-pqr}{1-q}\right\}\left\{-\frac{1-pq}{1-r}, \frac{1-pqr}{1-r}\right\} = 1$$

for $p, q, r \in \operatorname{rad} A$.

Consequently $K_2(n, A) \approx K_2(n + 1, A) \approx K_2(A)$.

Finally, suppose that $\mathfrak{O}$ is the ring of integers in an algebraic number field F and that $\mathfrak{p}$ is a maximal ideal of $\mathfrak{O}$ with $\mathfrak{p} \cap \boldsymbol{Z} = p\boldsymbol{Z}$. Put $e = e(\mathfrak{p}/p)$, the ramification index. Denote by $\hat{\mathfrak{O}}_\mathfrak{p}$ the completion of $\mathfrak{O}$ at $\mathfrak{p}$, with field of fractions $\hat{F}_\mathfrak{p}$, let $\hat{\mu}$ denote the roots of unity in $\hat{F}_\mathfrak{p}$, and let $\hat{\mu}_p$ be the p-primary component of $\hat{\mu}$. Moore ([Mo], [Mi, Theorem A.14]) has shown that $K_2(\hat{F}_\mathfrak{p}) \approx G \oplus \hat{\mu}$, where G is a divisible group.

COROLLARY. *$K_2(\hat{\mathfrak{O}}_\mathfrak{p}) \approx G \oplus \hat{\mu}_p$ where G is a divisible group.*

2. **Quotients of rings of integers.** We continue to use the notation of §1.

THEOREM 3. *$K_2(\mathfrak{O}/\mathfrak{p}^k)$ is a cyclic p-group of order p^t, where*

$$t = \left[\frac{k}{e} - \frac{1}{(p-1)}\right]_{[0,m]}, \qquad p^m = |\hat{\mu}_p|.$$

Here we write

$$[x]_{[0,m]} = \inf(\sup(0,[x]),m),$$

where $[x]$ denotes the greatest integer in x.

Since $\mathfrak{O}$ is a Dedekind domain and K_2 commutes with finite products, Theorem 3 allows us to compute $K_2(\mathfrak{O}/\mathfrak{a})$ for any ideal $\mathfrak{a} \subset \mathfrak{O}$. It should be noted that Theorem 3 implies the long conjectured result $K_2(\boldsymbol{Z}/2^n\boldsymbol{Z}) \approx \boldsymbol{Z}/2\boldsymbol{Z}$ for $n \geqq 2$.

There are three parts to the proof of Theorem 3. It is easily shown that $K_2(\mathfrak{O}/\mathfrak{p}^k)$ is a finite p-group for $k \geqq 1$. Since $K_2(\hat{\mathfrak{O}}_\mathfrak{p}) \to K_2(\mathfrak{O}/\mathfrak{p}^k)$ is surjective [St2, Theorem 2.13], the Corollary of §1 implies the existence of a surjection $\hat{\mu}_p \to K_2(\mathfrak{O}/\mathfrak{p}^k)$. Second, a topological argument using the norm residue symbol shows that for large values of k, there is a surjection $K_2(\mathfrak{O}/\mathfrak{p}^k) \to \hat{\mu}_p$. In the final part of the argument we determine exactly how the order of $K_2(\mathfrak{O}/\mathfrak{p}^k)$ can increase as k increases.

3. **Rings of integers.** The formula for the order of $K_2(\mathfrak{O}/\mathfrak{p}^k)$ given in Theorem 3 closely resembles that given by Bass-Milnor-Serre for the order of $SK_1(\mathfrak{O}, \mathfrak{p}^k)$ when $\mathfrak{O}$ is the ring of integers in a totally imaginary number field [B-M-S, Corollary 4.3c]. One difference, however, is that in our formula, p^m denotes the order of $\hat{\mu}_p$, the p-primary component of the roots of unity in $\hat{F}_\mathfrak{p}$, whereas in [B-M-S], p^m is the order of the p-primary component of the roots of unity in F itself. That these numbers are sometimes different may be exploited to yield several interesting examples.

Let $\mathfrak{O} = \boldsymbol{Z}[\sqrt{-17}]$ and let $\mathfrak{p} \subset \mathfrak{O}$ be a prime such that $\mathfrak{p}|2$. Then $\mathfrak{p}^2 = (2)$ and $\mathfrak{p}^6 = (8)$. Since $-17 \equiv -1$ modulo 16, $|\hat{\mu}_2| = 2^2$ [W, Proposition 6-5-5], whereas $\mathfrak{O}^* = \{\pm 1\}$. It thus follows from Theorem 3 and [B-M-S, Corollary 4.3c] that for $n \geqq 3$,

$$K_2(n, \mathfrak{O}/\mathfrak{p}^6) \approx \boldsymbol{Z}/4\boldsymbol{Z}, \qquad SK_1(\mathfrak{O}, \mathfrak{p}^6) \approx \boldsymbol{Z}/2\boldsymbol{Z}.$$

Using the exact sequence [Mi, Theorem 6.2]

$$K_2(n, \mathfrak{O}) \to K_2(n, \mathfrak{O}/\mathfrak{p}^6) \to SK_1(\mathfrak{O}, \mathfrak{p}^6) \to 0,$$

we conclude that there is a nonzero element $\sigma \in K_2(n, \mathfrak{O}/\mathfrak{p}^6)$ which lies in the image of $K_2(n, \mathfrak{O})$. But the only possibly nonzero symbol in $K_2(n, \mathfrak{O})$ is $\{-1, -1\}$, and modulo $\mathfrak{p}^6$ we have

$$\{-1, -1\} = \{(\sqrt{-17})^2, -1\} = 1.$$

Therefore σ is the image of an element of $K_2(n, \mathfrak{O})$ which is not a symbol. We conclude: *$K_2(n, \mathfrak{O})$ is not generated by symbols for any $n \geqq 3$.*

It has been shown [D] that the statements "$K_2(n, A)$ is generated by symbols" and "A is universal for GE_n" ([C, §2], [Si, §2]) are equivalent for commutative rings A. Thus $\mathfrak{O} = \boldsymbol{Z}[\sqrt{-17}]$ furnishes an example of a ring of integers which is not universal for GE_n if $n \geqq 3$.

Now since $\mathfrak{O}$ is not Euclidean, it follows from results of Cohn [C, §6 and Theorem 5.2] that $\mathfrak{O}$ *is* universal for GE_2 and, therefore, that $K_2(2, \mathfrak{O})$ is generated by symbols. Therefore *$K_2(2, \mathfrak{O}) \to K_2(n, \mathfrak{O})$ is not surjective for $n \geqq 3$.* This shows that the surjective stability theorem of [D] is the best possible result for a general ring of algebraic integers. It is, of course, possible to construct many similar examples by this procedure.

REFERENCES

[B-M-S] H. Bass, J. Milnor and J.-P. Serre, *Solution of the congruence subgroup problem for SL_n ($n \geqq 3$) and Sp^{2n} ($n \geqq 2$)*, Inst. Hautes Études Sci. Publ. Math. No. 33 (1967), 59–137. MR **39** #5574.

[B-T] H. Bass and J. Tate, *K_2 of global fields* (in preparation).

[C] P. M. Cohn, *On the structure of the GL_2 of a ring*, Inst. Hautes Études Sci. Publ. Math. No. 30 (1966), 5–53. MR **34** #7670.

[D] R. K. Dennis, *Surjective stability for the functor K_2* (to appear).

[M-K-S] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and Appl. Math. vol. 13, Interscience, New York, 1966. MR **34** #7617.

[Ma] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62. MR **39** #1566.

[Mi] J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Studies, Princeton Univ. Press, Princeton, N.J., 1971.

[Mo] C. Moore, *Group extensions of p-adic and adelic linear groups*, Inst. Hautes Études Sci. Publ. Math. No. 35 (1968), 5–70. MR **39** #5575.

[Si] J. Silvester, *A presentation of GL_n ($\boldsymbol{Z}$) and GL_n ($k[X]$)* (to appear).

[St1] M. Stein, *Generators, relations and coverings of Chevalley groups over commutative rings*, Amer. J. Math. **93** (1971), 965–1004.

[St2] ———, *Surjective stability in dimension 0 for K_2 and related functors* (to appear).

[W] E. Weiss, *Algebraic number theory*, McGraw-Hill, New York, 1963. MR **28** # 3021.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14850

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201