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NONSEPARATING FUNCTION ALGEBRAS

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Let A be a function algebra on X (compact). We say A is a separating algebra on X if for each closed subset S of X and for each $x \in X \backslash S$ there exists f in A such that $f(x) = 0$ and f does not vanish on S. We say that A is essential on X if for each open subset U of X there is a continuous function $f \notin A$ such that f vanishes on X/U. Csordas and Reiter asked [**2**] if there exists a nonseparating, essential algebra A on a (connected) space X for which X is the maximal ideal space of A and also the Šilov boundary of A. We give an example of such an algebra and simple examples of nonseparating algebras.

Given a compact subset K of C^n, let $P(K)$ denote the uniform closure in $C(K)$ of the polynomials in $z_1, \ldots, z_n$. An easy application of Hurwitz' theorem [**1**, p. 176] shows that the first three of the following algebras are nonseparating.

EXAMPLE 1. Let $\Delta = \{z : |z| \leqq 1\}$. Then $P(\Delta \times \Delta)$ is nonseparating since $f(\Delta \times \Delta) = f(\{(z, w) : |z| = 1 \text{ or } |w| = 1\})$ for each f in $P(\Delta \times \Delta)$. Also, $\Delta \times \Delta$ is the maximal ideal space of $P(\Delta \times \Delta)$.

EXAMPLE 2. The algebra $P([0, 1] \times \Delta)$ is nonseparating since $f([0, 1] \times \Delta) = f(\{(t, z) : t = 0 \text{ or } |z| = 1\})$. Also, $[0, 1] \times \Delta$ is the maximal ideal space of $P([0, 1] \times \Delta)$.

EXAMPLE 3. If A is a separating algebra on $X = M(A)$ and if B is a function algebra on X containing A, then B is separating on X but not necessarily separating on $M(B)$. Let B denote the uniform closure in $C(\Delta)$ of polynomials in z and $|z|$. Then $P(\Delta) \subseteq B \subseteq C(\Delta)$. One can embed B into $P([0, 1] \times \Delta)$ by setting $F(r, z) = f(rz)$ for $0 \leqq r \leqq 1$ and $|z| \leqq 1$. Now one can see that the maximal ideal space of B is $[0, 1] \times \Delta/\{0\} \times \Delta$ and so B is nonseparating on its maximal ideal space.

EXAMPLE 4. Let $S^2 = C \cup \{\infty\}$ and let $D = \{z : |z| < 1\}$. Let

$$B = \{f \in C(S^2) : f \text{ is analytic in } D\}.$$

Then $M(B) = S^2$ and $f(S^2) = f(S^2 \backslash D)$ for each f in B. One easily checks that S^2 is the maximal ideal space of B. Fix f in B and assume f does not vanish on $S^2 \backslash D$. If f has a zero in D, then there exist $\alpha_1, \ldots, \alpha_n \in D$ such that $g(z) = f(z) \prod_{k=1}^{n} z/(z - \alpha_k)$ belongs to B and vanishes only at 0. Given $\infty \geqq r > 0$, define $\Gamma_r(\theta) = g(re^{i\theta})$. All of the curves Γ_r are homotopic

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to each other in $C\backslash\{0\}$ and Γ_∞ is constant. For r small, the winding number of Γ_r around 0 is equal to the order of the zero of g at 0. Hence, $f(S^2) = f(S^2\backslash D)$.

EXAMPLE 5. We now construct an algebra A on a space X such that A is nonseparating and essential and X is the maximal ideal space and Šilov boundary of A. (1) Choose $z_i \in C$ and $r_i > 0$ such that $\sum r_i < \infty$, the discs $D_i = \{z:|z - z_i| < r_i\}$ are mutually disjoint and contained in $\Delta = \{z:|z| \leqq 1\}$, and $Y = \Delta\backslash \bigcup_{i=1}^{\infty} D_i$ has no interior. (2) Let $R(Y)$ denote the uniform closure in $C(Y)$ of the rational functions with poles off Y. Then $R(Y)$ is essential and Y is the maximal ideal space and the Šilov boundary of $R(Y)$. (3) Let B be a nonseparating function on a compact metric space K' where $K' = M(B)$ [Examples 1–4]. Let K be a Cantor set of measure zero in $\{z:|z| = 1\}$ and let ρ be a continuous map of K onto K'. Now set $A = \{f \in R(Y):f \in B \circ \rho\}$. Since K is a peak interpolation set for $R(Y)$, the space Y/ρ is the maximal space of A. (See [**3**, Remark 2.2].) A is essential and Y/ρ is the Šilov boundary of A. Finally, A is nonseparating.

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