ABSTRACT. Let \( E \) be a set of positive measure on the unit circle. Let \( f \in H^p (1 \leq p \leq \infty) \) and \( g \) be the restriction of \( f \) to \( E \). It is shown that functions \( g_\lambda, \lambda > 0 \), can be constructed from \( g \) so that \( g_\lambda \to f \). We also characterize those functions \( g \) on \( E \) which are restrictions of functions in \( H^p (1 < p \leq \infty) \).

In the following, the space \( H^p (1 \leq p \leq \infty) \) will, according to the context, be either the Hardy class of analytic functions in the open unit disc \( D \) or the space of the corresponding boundary value functions, viz the subspace of “analytic” functions in \( L^p(C) \), \( C \) being the unit circle. If \( E \subset C \) has positive measure then it is well known (see [3]) that a function in \( H^p \) cannot vanish on \( E \) without being identically zero. Thus, theoretically at least, \( f \in H^p \) is uniquely “determined” by its values on \( E \). In the present work we address ourselves to the problem of recovering functions in \( H^p \) from their restrictions to \( E \). Theorem I gives an explicit constructive solution to this problem. The allied problem of characterizing the restrictions to \( E \) of functions in \( H^p (1 < p \leq \infty) \) is solved in Theorem II. To the best of our knowledge, the only known results relating to these problems are due to the author [4] where the case \( p = 2 \) is dealt with.

**Theorem I.** Let \( E \subset C \) with \( m(E) > 0 \). Suppose that \( 1 \leq p \leq \infty \), \( f \in H^p \) and that \( g \) is the restriction of \( f \) to \( E \). For each \( \lambda > 0 \) define analytic functions \( h_\lambda, g_\lambda \) on \( D \) by

\[
\begin{align*}
h_\lambda(z) &= \exp\left\{- \frac{1}{4\pi} \log(1 + \lambda) \int_E e^{i\theta} + z \overline{e^{i\theta} - z} \, d\theta \right\}, \quad z \in D, \\
g_\lambda(z) &= \lambda h_\lambda(z) \frac{1}{2\pi i} \int_E \frac{\overline{h_\lambda(w)}g(w) \, dw}{w - z}, \quad z \in D.
\end{align*}
\]

Then as \( \lambda \to \infty \), \( g_\lambda \to f \) uniformly on compact subsets of \( D \). Moreover for \( 1 < p < \infty \) we also have \( \|g_\lambda - f\|_p \to 0 \) as \( \lambda \to \infty \).

**Theorem II.** Let \( E \subset C \) with \( 0 < m(E) < m(C) \). For \( g \in L^1(E) \) let \( g_\lambda \) be as in Theorem I. (a) If \( 1 < p < \infty \) then a function \( g \in L^p(E) \) is the restriction to \( E \) of some \( f \in H^p \) if and only if \( \sup_{\lambda > 0} \|g_\lambda\|_p < \infty \). (b) A function \( g \in L^\infty(E) \) is the restriction to \( E \) of some \( f \in H^\infty \) if and only if \( \sup_{p > 1} \lim \sup_{\lambda \to \infty} \|g_\lambda\|_p < \infty \).
The proof of Theorem I will be based on a series of lemmas. First we recall some elementary properties of Toeplitz operators on $H^p$ spaces (for details in the special case $p = 2$ see [1], and for the general case $1 < p < \infty$ see [5]). Let $1 < p < \infty$. For each $\phi \in L^\infty$, the Toeplitz operator $T_\phi$ is defined by $T_\phi f = P(\phi f)$, $f \in H^p$, where $P$ is the natural projection of $L^p$ onto $H^p$. We need the following facts: (i) $\|T_\phi\| \leq C_{\phi}\|\phi\|_\infty$, (ii) if $\phi, \psi \in L^\infty$ and if either $\phi \in H^\infty$ or $\psi \in H^\infty$, then $T_\phi f = T_\psi f$. This latter fact immediately yields

**Lemma 1.** If $h, 1/h \in H^\infty$ and $\varphi = |h|^{-2}$, then the Toeplitz operator $T_\varphi$ is invertible and $T_\varphi^{-1} = T_h$. \[ \text{Proof.} \quad T_h T_\varphi T_h = T_h (T_h T_{1/h}) T_{1/h} = T_h T_{1/h} = I, \text{ etc.} \]

Let $\chi_E$ be the characteristic function of the set $E$ and let for $\lambda > 0$, $\varphi_\lambda = 1 + \lambda \chi_E$. Then the function $h_\lambda$ defined in Theorem I satisfies, $1/\varphi_\lambda = h_\lambda h_{1/\lambda}$. Also $h_\lambda, 1/h_\lambda \in H^\infty$. Thus by Lemma 1, we have

**Lemma 2.** $T_{\varphi_\lambda}$ is invertible and $T_{\varphi_\lambda}^{-1} = T_{h_\lambda} T_{h_{1/\lambda}}$.

**Lemma 3.** Define for each $a \in D$, $e_a(z) = 1/(1 - az), z \in D$. Then $e_a \in H^p$, $1 \leq p \leq \infty$, and if $T_{e_a}$ is treated as an operator on $H^p$ ($1 < p < \infty$), we have $T_{e_a}^{-1} e_a = h_\lambda(a) h_{1/\lambda} e_a$. \[ \text{Proof.} \quad \text{For each } g \in H^q \quad (q = p/(p - 1)), \quad \text{we have} \quad (T_{h_\lambda} e_a, g) = (e_a, h_\lambda g) = h_\lambda(a) (h_\lambda(a) e_a, g). \]

Thus $T_{h_\lambda} e_a = h_\lambda(a) e_a$. An appeal to Lemma 2 finishes the proof.

**Lemma 4.** Let $K$ be a compact subset of $D$ and $1 \leq p \leq \infty$. Then as $\lambda \to \infty$, $\|h_\lambda(a) h_{1/\lambda} e_a\|_p \to 0$ uniformly for $a \in K$. \[ \text{Proof.} \quad \text{We note that} \quad \|h_{1/\lambda}\|_\infty \leq 1 \quad \text{and} \quad \|h_\lambda(a)\| \leq (1 + \lambda)^{-\alpha} \text{ where } \alpha > 0 \quad \text{and} \quad \alpha \text{ depends on } |a|. \]

Let now $S$ be the Toeplitz operator on $H^p$ ($1 < p < \infty$) corresponding to the characteristic function $\chi_E$ of $E$. Then since $I + \lambda S = T_{\varphi}$, $(I + \lambda S)^{-1}$ exists by Lemma 2. Also by Lemma 4, $\|(I + \lambda S)^{-1} e_a\|_p \to 0$ as $\lambda \to \infty$. By Lemma 2 and fact (i) about Toeplitz operators we also have

$$\|(I + \lambda S)^{-1}\| = \|T_{h_{1/\lambda}}\| \leq \|h_{1/\lambda}\|^2 C_p \leq C_p.$$  

Noting that $\{e_a : a \in D\}$ is a fundamental set in $H^p$, we therefore obtain (cf., e.g., [3, p. 55]) that $\|(I + \lambda S)^{-1} f\|_p \to 0$ for every $f \in H^p$. Noting that for $f \in H^p$, $(I + \lambda S)^{-1} f = f - \lambda (I + \lambda S)^{-1} Sf$, we get

**Lemma 5.** If $1 < p < \infty$ and $f \in H^p$, then as $\lambda \to \infty$, $\|\lambda (I + \lambda S)^{-1} Sf - f\|_p \to 0$. \[ \text{Proof.} \quad \text{Let} \quad \lambda (I + \lambda S)^{-1} Sf - f \quad \text{be the error.} \]

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The proof of Theorem I (for $1 < p < \infty$) will be complete if we show that $g_\lambda = \lambda(I + \lambda S)^{-1}Sf$. This is routine: For $z \in D$,

$$(\lambda(I + \lambda S)^{-1}Sf, e_z) = \lambda(Sf, (I + \lambda S)^{-1}e_z) = \lambda(\chi_E, f, (I + \lambda S)^{-1}e_z) = \lambda(f, e_z) = \lambda(f, h_\lambda(z)h_\lambda e_z)_E.$$

In the above chain of equalities, the first is a consequence of the fact that $(I + \lambda S)^*$ is the operator $(I + \lambda S)$ on $H^q (q = p/(p - 1))$ and the last results from Lemma 3. The notation $(\, , )_E$ denotes the “inner product” over the set $E$. Now it can be readily checked that $\lambda(f, h_\lambda(z)h_\lambda e_z)_E$ is the same as the defining expression for $g_\lambda(z)$.

The case $p = \infty$ is easy. If $f \in H^\infty$ then since $f$ is also in $H^2$, by the preceding, $\|g_\lambda - f\|_2 \to 0$ and hence $g_\lambda \to f$ uniformly on compact subsets of $D$.

Turning to the case $p = 1$, let $f \in H^1$. For $0 < r < 1$, define $f_r$ by $f_r(e^{i\theta}) = f(re^{i\theta})$. Then as is well known, $\|f_r\|_1 \leq \|f\|_1$ and $\|f_r - f\|_1 \to 0$ as $r \to 1$. Let us define, for each $\lambda > 0$, $f_{r,\lambda}$ by

$$f_{r,\lambda}(z) = \frac{1}{2\pi i} \int_E \frac{h_\lambda(w)f_r(w)}{w - z} dw, \quad z \in D.$$ 

Then we see that, for every compact set $K \subset D$, the following statements hold uniformly in $K$: (1) $f_{r,\lambda} \to g_\lambda$ as $r \to 1$, (2) $f_r \to f$ as $r \to 1$, (3) $f_{r,\lambda} \to f_r$ as $\lambda \to \infty$. The less trivial of these statements, viz. (3), follows because $f_r \in H^2$ and the case $p = 2$ of the theorem applies. If we show further that the convergence in (3) is also uniform for $r$ in $(0, 1)$ then we can conclude that $g_\lambda \to f$ as $\lambda \to \infty$ uniformly in $K$ and the proof of the theorem for $p = 1$ will be complete. For this purpose, remembering that $f \in H^2$ we have for each $z \in K$,

$$f_{r,\lambda}(z) - f_r(z) = (\lambda(I + \lambda S)^{-1}Sf_r - f_r, e_z) = (I + \lambda S)^{-1}f_r, e_z) = (f_r, (I + \lambda S)^{-1}e_z) = (f_r, h_\lambda(z)h_\lambda e_z).$$

Hence we obtain

$$|f_{r,\lambda}(z) - f_r(z)| \leq \|f_r\|_1 \|h_\lambda(z)h_\lambda e_z\|_\infty \leq \|f\|_1 \|h_\lambda(z)h_\lambda e_z\|_\infty.$$ 

The last term is independent of $r$ and Lemma 4 ($p = \infty$) does the job.

**PROOF OF THEOREM II.** The “only if” parts are evident from Theorem I.

As for the “if” part in (a), the boundedness of $\{\|g_\lambda\|_p\}$ together with the weak* compactness of closed balls in $H^p$ provide us with a sequence $\lambda_n \to \infty$ such that $g_{\lambda_n}$ converges weak* to some $f$ in $H^p$. Let $g_1 \in L^p(C)$ be defined by setting $g_1 = g$ on $E$ and $g_1 = 0$ otherwise. Denote $Pg_1$ by $\tilde{g}$. From the discussion following Lemma 5, it can be seen that

$$g_\lambda = \lambda(I + \lambda S)^{-1}\tilde{g}.$$
Thus for every $k \in H^q$ ($q = p/(p - 1)$), $(\lambda_n(I + \lambda_n S)^{-1}S\hat{g}, k) = (g_{\lambda_n}, Sk) \rightarrow (Sf, k) = (Sf, k)$, while by Lemma 5, the first of these inner products converges to $(\hat{g}, k)$. Hence $\hat{g} = Sf$. This means that the Fourier coefficients $((f - g_i)\chi_E)(n)$ are zero for $n \geq 0$. In other words, $((f - g_i)\chi_E) \in H^p$.

Since $m(C \Delta E) > 0$, we must have $f = g_{\chi_E}$. For proving the "if" part in (b) we need to make just two observations. First, $g \in L^\infty(E)$ implies $g_{\lambda} \in H^p$ for each $p < \infty$ and hence part (a) gives $f$ belonging to $H^p$ for all $p < \infty$ and such that $g$ is the restriction to $E$ of $f$. Secondly, $\|g_{\lambda}\|_p \rightarrow \|f\|_p$ as $\lambda \rightarrow \infty$ and $\|f\|_p \rightarrow \|f\|_\infty$ as $p \rightarrow \infty$. The details are left to the reader.

**Remarks.** 1. In the proof of Theorem I, we did not use the F. & M. Riesz Theorem. We thus obtain a new proof of the statement: if $f \in H^p$ ($1 \leq p \leq \infty$), $f = 0$ on $E$, $m(E) > 0$, then $f = 0$.

2. Theorem I points out a way which enables us to draw conclusions about the properties of a holomorphic function from the knowledge of its values on an arc. It is possible to obtain results parallel to the classical Cauchy theory where we now have integrals over a curve which may not be closed. Details of these and other related results will be published elsewhere.

**References**


