ABSTRACT. Let $E$ be a set of positive measure on the unit circle. Let $f \in H^p (1 \leq p \leq \infty)$ and $g$ be the restriction of $f$ to $E$. It is shown that functions $g_\lambda$, $\lambda > 0$, can be constructed from $g$ so that $g_\lambda \to f$. We also characterize those functions $g$ on $E$ which are restrictions of functions in $H^p (1 < p \leq \infty)$.

In the following, the space $H^p (1 \leq p \leq \infty)$ will, according to the context, be either the Hardy class of analytic functions in the open unit disc $D$ or the space of the corresponding boundary value functions, viz. the subspace of "analytic" functions in $L^p (C)$, $C$ being the unit circle. If $E \subset C$ has positive measure then it is well known (see [3]) that a function in $H^p$ cannot vanish on $E$ without being identically zero. Thus, theoretically at least, $f \in H^p$ is uniquely "determined" by its values on $E$. In the present work we address ourselves to the problem of recovering functions in $H^p$ from their restrictions to $E$. Theorem I gives an explicit constructive solution to this problem. The allied problem of characterizing the restrictions to $E$ of functions in $H^p (1 < p \leq \infty)$ is solved in Theorem II. To the best of our knowledge, the only known results relating to these problems are due to the author [4] where the case $p = 2$ is dealt with.

**Theorem I.** Let $E \subset C$ with $m(E) > 0$. Suppose that $1 \leq p \leq \infty$, $f \in H^p$ and that $g$ is the restriction of $f$ to $E$. For each $\lambda > 0$ define analytic functions $h_\lambda, g_\lambda$ on $D$ by

$$h_\lambda(z) = \exp\left\{-\frac{1}{4\pi} \log(1 + \lambda) \int_D \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta\right\}, \quad z \in D,$$

$$g_\lambda(z) = \lambda h_\lambda(z) \frac{1}{2\pi i} \int_E \frac{\overline{h_\lambda(w)} g(w) \, dw}{w - z}, \quad z \in D.$$  

Then as $\lambda \to \infty$, $g_\lambda \to f$ uniformly on compact subsets of $D$. Moreover for $1 < p < \infty$ we also have $\|g_\lambda - f\|_p \to 0$ as $\lambda \to \infty$.

**Theorem II.** Let $E \subset C$ with $0 < m(E) < m(C)$. For $g \in L^1(E)$ let $g_\lambda$ be as in Theorem I. (a) If $1 < p < \infty$ then a function $g \in L^p(E)$ is the restriction to $E$ of some $f \in H^p$ if and only if $\sup_{\lambda > 0} \|g_\lambda\|_p < \infty$. (b) A function $g \in L^\infty(E)$ is the restriction to $E$ of some $f \in H^\infty$ if and only if $\sup_{p > 1} \limsup_{\lambda \to \infty} \|g_\lambda\|_p < \infty$.

**AMS 1970 subject classifications.** Primary 30A78, 30A88; Secondary 46E20, 30A86, 47B35.

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The proof of Theorem I will be based on a series of lemmas. First we recall some elementary properties of Toeplitz operators on $H^p$ spaces (for details in the special case $p = 2$ see [1], and for the general case $1 < p < \infty$ see [5]). Let $1 < p < \infty$. For each $\varphi \in L^\infty$, the Toeplitz operator $T_\varphi$ is defined by $T_\varphi f = P(\varphi f)$, $f \in H^p$, where $P$ is the natural projection of $L^p$ onto $H^p$. We need the following facts: (i) $\|T_\varphi\| \leq C_p \|\varphi\|_\infty$, (ii) if $\varphi, \psi \in L^\infty$ and if either $\varphi \in H^\infty$ or $\psi \in H^\infty$, then $T_\varphi \psi = T_\varphi T_\psi$. This latter fact immediately yields

**Lemma 1.** If $h, l \in H^\infty$ and $q > h \sim \frac{1}{2}$, then the Toeplitz operator $T_\varphi$ is invertible and $T_\varphi^{-1} = T_{\varphi^{-1}}$.

**Proof.** $T_\varphi T_\varphi^{-1} = T_{\varphi^{-1}} = T_{\varphi^{-1}} = \mathbf{I}$, etc.

Let $\chi_E$ be the characteristic function of the set $E$ and let for $\lambda > 0$, $\varphi_\lambda = 1 + \lambda \chi_E$. Then the function $h_\lambda$ defined in Theorem I satisfies, $1/\varphi_\lambda = h_\lambda h_\lambda$. Also $h_\lambda, 1/\varphi_\lambda \in H^\infty$. Thus by Lemma 1, we have

**Lemma 2.** $T_{\varphi_\lambda}$ is invertible and $T_{\varphi_\lambda}^{-1} = T_{h_\lambda} T_{\varphi_\lambda}^{-1}$.

**Lemma 3.** Define for each $a \in D$, $e_a(z) = 1/(1 - az)$, $z \in D$. Then $e_a \in H^p$, $1 \leq p \leq \infty$, and if $T_{e_a}$ is treated as an operator on $H^p$ ($1 < p < \infty$), we have $T_{e_a}^{-1} e_a = h_\lambda(a) h_\lambda e_a$.

**Proof.** For each $g \in H^q$ ($q = p/(p - 1)$), we have $(T_{e_a} g, h_\lambda)$ $= (e_a h_\lambda, g) = h_\lambda(a) g(a) = h_\lambda(a) (e_a, g)$. Thus $T_{e_a} e_a = h_\lambda(a) e_a$. An appeal to Lemma 2 finishes the proof.

**Lemma 4.** Let $K$ be a compact subset of $D$ and $1 \leq p \leq \infty$. Then as $\lambda \to \infty$, $\|h_\lambda(a) h_\lambda e_a\|_p \to 0$ uniformly for $a \in K$.

**Proof.** We note that $\|h_\lambda\|_\infty \leq 1$ and $\|h_\lambda(a)\| \leq (1 + \lambda)^{-\alpha}$ where $\alpha > 0$ and $\alpha$ depends on $|a|$.

Let now $S$ be the Toeplitz operator on $H^p$ ($1 < p < \infty$) corresponding to the characteristic function $\chi_E$ of $E$. Then since $I + \lambda S = T_{\varphi_\lambda}$, $(I + \lambda S)^{-1}$ exists by Lemma 2. Also by Lemma 4, $\|(I + \lambda S)^{-1} e_a\|_p \to 0$ as $\lambda \to \infty$. By Lemma 2 and fact (i) about Toeplitz operators we also have

$$\|(I + \lambda S)^{-1}\| = \|T_{h_\lambda} T_{\varphi_\lambda}^{-1}\| \leq \|h_\lambda\|_\infty^2 C_p^2 \leq C_p^2.$$ 

Noting that $\{e_a : a \in D\}$ is a fundamental set in $H^p$, we therefore obtain (cf., e.g., [3, p. 55]) that $\|(I + \lambda S)^{-1} f\|_p \to 0$ for every $f \in H^p$. Noting that for $f \in H^p$, $(I + \lambda S)^{-1} f = f - \lambda(I + \lambda S)^{-1} S f$, we get

**Lemma 5.** If $1 < p < \infty$ and $f \in H^p$, then as $\lambda \to \infty$,

$$\|\lambda(I + \lambda S)^{-1} S f - f\|_p \to 0.$$

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The proof of Theorem I (for $1 < p < \infty$) will be complete if we show that $g_\lambda = \lambda(I + \lambda S)^{-1}Sf$. But this is routine: For $z \in D$,
\[
(\lambda(I + \lambda S)^{-1}Sf, e_z) = \lambda(Sf, (I + \lambda S)^{-1}e_z) = \lambda(\chi_E f, (I + \lambda S)^{-1}e_z) = \lambda(f, (I + \lambda S)^{-1}e_z)_E = \lambda(f, \bar{h}_\lambda(z)h_\lambda e_z)_E.
\]
In the above chain of equalities, the first is a consequence of the fact that $(I + \lambda S)^{-1}$ is the operator $(I + \lambda S)$ on $H^q$ ($q = p/(p - 1)$) and the last results from Lemma 3. The notation $(\ , \ , )_E$ denotes the “inner product” over the set $E$. Now it can be readily checked that $\lambda(f, \bar{h}_\lambda(z)h_\lambda e_z)_E$ is the same as the defining expression for $g_\lambda(z)$.

The case $p = \infty$ is easy. If $f \in H^\infty$ then since $f$ is also in $H^2$, by the preceding, $\|g_\lambda - f\|_2 \to 0$ and hence $g_\lambda \to f$ uniformly on compact subsets of $D$.

Turning to the case $p = 1$, let $f \in H^1$. For $0 < r < 1$, define $f_r$ by $f_r(e^{i\theta}) = f(re^{i\theta})$. Then as is well known, $\|f_r\|_1 \leq \|f\|_1$ and $\|f_r - f\|_1 \to 0$ as $r \to 1$. Let us define, for each $\lambda > 0$, $f_{r,\lambda}$ by
\[
f_{r,\lambda}(z) = \lambda h_\lambda(z) \frac{1}{2\pi i} \int_E \bar{h}_\lambda(w)f_r(w) \frac{w - z}{w} \, dw, \quad z \in D.
\]
Then we see that, for every compact set $K \subset D$, the following statements hold uniformly in $K$: (1) $f_{r,\lambda} \to g_\lambda$ as $r \to 1$, (2) $f_r \to f$ as $r \to 1$, (3) $f_{r,\lambda} \to f_r$ as $\lambda \to \infty$. The less trivial of these statements, viz. (3), follows because $f_r \in H^2$ and the case $p = 2$ of the theorem applies. If we show further that the convergence in (3) is also uniform for $r$ in $(0, 1)$ then we can conclude that $g_\lambda \to f$ as $\lambda \to \infty$ uniformly in $K$ and the proof of the theorem for $p = 1$ will be complete. For this purpose, remembering that $f \in H^2$ we have for each $z \in K$,
\[
f_{r,\lambda}(z) - f_r(z) = (\lambda(I + \lambda S)^{-1}Sf_r - f_r, e_z) = ((I + \lambda S)^{-1}f_r, e_z) = (f_r, (I + \lambda S)^{-1}e_z) = (f_r, \bar{h}_\lambda(z)h_\lambda e_z).
\]
Hence we obtain
\[
|f_r(z) - f_{r,\lambda}(z)| \leq \|f_r\|_1 \|\bar{h}_\lambda(z)h_\lambda e_z\|_\infty \leq \|f\|_1 \|\bar{h}_\lambda(z)h_\lambda e_z\|_\infty.
\]
The last term is independent of $r$ and Lemma 4 ($p = \infty$) does the job.

**Proof of Theorem II.** The “only if” parts are evident from Theorem I. As for the “if” part in (a), the boundedness of $\{\|g_\lambda\|_p\}$ together with the weak* compactness of closed balls in $H^p$ provide us with a sequence $\lambda_n \to \infty$ such that $g_{\lambda_n}$ converges weak* to some $f$ in $H^p$. Let $g_1 \in L^p(C)$ be defined by setting $g_1 = g$ on $E$ and $g_1 = 0$ otherwise. Denote $P g_1$ by $\hat{g}$. From the discussion following Lemma 5, it can be seen that
\[
g_\lambda = \lambda(I + \lambda S)^{-1}\hat{g}.
\]
Thus for every $k \in H^q$ ($q = p/(p - 1)$), $(\lambda_n(I + \lambda_nS)^{-1}S\hat{g}, k) = (g_{\lambda_n}, Sk) \to (f, Sk) = (Sf, k)$, while by Lemma 5, the first of these inner products converges to $(\hat{g}, k)$. Hence $\hat{g} = Sf$. This means that the Fourier coefficients $((f - g_1)x_E(n)$ are zero for $n \geq 0$. In other words, $(f - g_1)x_E \in H^p$. Since $m(C \setminus E) > 0$, we must have $f = g_1$ on $E$.

For proving the "if" part in (b) we need to make just two observations. First, $g \in L^\infty(E)$ implies $g_{\lambda} \in H^p$ for each $p < \infty$ and hence part (a) gives $f$ belonging to $H^p$ for all $p < \infty$ and such that $g$ is the restriction to $E$ of $f$. Secondly, $\|g_{\lambda}\|_p \to \|f\|_p$ as $\lambda \to \infty$ and $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$. The details are left to the reader.

**Remarks.** 1. In the proof of Theorem I, we did not use the F. & M. Riesz Theorem. We thus obtain a new proof of the statement: if $f \in H^p$ ($1 \leq p \leq \infty$), $f = 0$ on $E$, $m(E) > 0$, then $f = 0$. 2. Theorem I points out a way which enables us to draw conclusions about the properties of a holomorphic function from the knowledge of its values on an arc. It is possible to obtain results parallel to the classical Cauchy theory where we now have integrals over a curve which may not be closed. Details of these and other related results will be published elsewhere.

**References**


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