THE SPECTRUM OF AN AUTOMORPHISM

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In a series of articles H. Kamowitz and I investigated the nature of $\sigma(T)$, the spectrum of an arbitrary automorphism of an arbitrary semisimple commutative Banach algebra. This study was begun as a by-product of \[1\], in which we made the incidental observation that $\sigma(T)$ must meet \(\{z:|z-1| \geq 1\}\), unless $T = I$. The following is a summary of the known necessary conditions (N) and the known sufficient conditions (S) on $\sigma(T)$.

**N1.** If $T^k = I$ (some $k \geq 1$), then $\sigma(T)$ is a union of subgroups of the group of $k$th roots of 1, \[2\].

**S1.** Every possibility consistent with N1 can occur (direct sums of rotations).

**N2.** If $T^k \neq I$ (all $k \geq 1$), then $\sigma(T)$ is the unit circle, \[2\].

**S2.** It is common that $\sigma(T)$ is the unit circle, but $\sigma(T)$ can be an annulus, \[2\].

**N3.** If $T^k \neq I$ (all $k \geq 1$), then $\sigma(T)$ must be connected, \[3\].

**S3.** The set of $\sigma(T)$’s is closed under the mapping $1/z$, and if $U = \bigcup \sigma(T_k)$ is bounded away from 0 and $\infty$, then $\overline{U}$ is $\sigma(T)$ for some $T$. If $R$ is a bounded region such that $\{1 < |z| < a\} \subseteq R \subseteq \{1 < |z|\}$ and $\{1 < |z|\} - R$ is a semigroup under multiplication, then $\overline{R}$ is $\sigma(T)$ for some $T$. The hypothesis that $R$ be connected may be weakened somewhat, \[3\].

The purpose of this note is to extend the set of constructions of \[3\] to include cases where $\sigma(T)$ is not the closure of its interior. The following theorem illustrates the technique of attaching a line segment to a region.

**THEOREM.** Let $\sigma = \{z:1 \leq |z| \leq 2\} \cup \{z:2 \leq z \leq 3\}$. Then there is a semisimple Banach algebra $A$ and an automorphism $T$ of $A$ such that $\sigma(T) = \sigma$.

**PROOF.** In the outline which follows I have omitted several routine calculations. Let $A$ be the set of all functions which are bounded and analytic on $\{1 < |z| < 2\}$ and $C^\infty$ on $\{1.5 \leq z \leq 3\}$ and satisfy $|f^{(n)}(z)| \leq B \max(1,n!(\log n)^n)$ for some $B < \infty$, all $n \geq 0$, and $1.5 \leq z \leq 3$. Define $p(f) = \sup\{|f(z)|:1 < |z| < 2\} + \inf B$. It is clear that $p$ is a norm for $A$ and that $A$ is complete with respect to $p$.

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Define \( f \ast g = \sum_{a_n b_n z^n} \), where \( f = \sum a_n z^n \) and \( g = \sum b_n z^n \). When \( f \) and \( g \) belong to \( A \), \( f \ast g \) is analytic on \( 1 < |z| < 4 \) and

\[
 f \ast g(z) = \frac{1}{2\pi i} \int_{|w| = 1} f(w) g\left( \frac{z}{w} \right) \frac{dw}{w} \quad \text{for } 1 < |z| < 2.
\]

It follows that \( f \ast g \in A \) and \( p(f \ast g) \leq \text{const } p(f) \cdot p(g) \). Then \( \|f\| = \text{const } p(f) \) defines a Banach algebra norm on \( A \).

The mapping \( f \to a_n \) is a homomorphism of \( A \) onto \( C \) for each \( n \). If \( a_n = 0 \) for all \( n \), then \( f \equiv 0 \); this is obvious for \( 1 < |z| < 2 \); for \( 1.5 \leq z \leq 3 \) it is a consequence of Carleman’s theorem on quasi-analytic classes [4, Chapter 1], since the \( n \)-th root of \( n! (\log n)^n \) is asymptotic to \( (n/e \log n) \).

Thus, \( A \) is semisimple.

Because of the rapid growth of \( n! (\log n)^n \), every function which is analytic on a neighborhood of \( \sigma \) belongs to \( A \). Furthermore, if \( g \) is such a function and \( f \) is arbitrary in \( A \), then \( g f \in A \) and \( \|g f\| \leq \text{const } \|f\| \).

Define \( T : A \to A \) by \( T f(z) = z f(z) \). \( T \) is an automorphism of \( A \) and \( \sigma(T) \supseteq \sigma \). If \( \lambda \notin \sigma \), use \( g = 1/(z - \lambda) \) in the preceding paragraph and we see that \( \sigma(T) = \sigma \).

**Remark.** The construction given above can be extended. As an illustration let us attach a new line segment to the old one. For example, let \( \sigma' = \sigma \cup \{ z : z = 3 + iy, 0 \leq y \leq 1 \} \). Define \( A' \) to be all functions which are bounded and analytic on \( \{ 1 < |z| < 2 \} \), \( C^\infty \) on each interval \( \{ 1.5 \leq z \leq 3 \} \) and \( \{ 3 + iy : 0 \leq y \leq 1 \} \) with \( |f^{(0)}| \leq B \max(1, n! (\log n)^n) \) on both intervals, and satisfying the Cauchy-Riemann condition \( (\partial / \partial x)^n f = (i^{-1} \partial / \partial y)^n f \) at \( z = 3 \). The rest of the proof continues now with very slight changes. (Observe that the Cauchy-Riemann condition guarantees that any function analytic on a neighborhood of \( \sigma' \) will belong to \( A' \) and that any member of \( A' \) which is 0 on \( \sigma \) will be 0 on \( \sigma' \).)

With the method of the theorem, disjoint domains can be connected by line segments, subject to the semigroup requirement of S3, and these constructions may be combined with those of [3] and iterated to produce quite complicated \( \sigma(T) \).

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**References**


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