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BOOK REVIEWS

Best approximation in normed linear spaces by elements of linear subspaces, by Ivan Singer. Die Grundlehren der mathematischen Wissenschaften, Band 171, Publishing House of the Academy of the Socialist Republic of Romania, Bucharest, 1970 and Springer-Verlag, New York-Heidelberg-Berlin, 1970. 415 pp., $21.50.

This monograph is a translation (by Radu Georgescu) of the original Romanian version "Cea mai bună aproximareon spaţii vectoriale normate prin elemente din subspaţii vectoriale" that was published in 1967. In spite of its language the original was widely used and frequently quoted (cf., e.g., [**9**] and [**10**]) by approximation theory people over the world, proving that the demand for this sort of work did exist. There appeared two reviews of it [**1**], [**2**] and the second one made the point that "it is unfortunate that a book which has been compiled with much work and care should have no terminological index, no author's index and no index of notations." This remark has been taken into account by the editors of the present edition, and we have obtained all of these indices now. Beside this addition, there are no changes except corrections of minor misprints and errors. Strangely enough, the false (as it was pointed out in [**1**]) equality before the formula (4.19) Chapter I remained unchanged.

The book deals with this part of the theory described in the title which has the clear-cut geometric interpretation. It is addressed to a large circle of mathematicians ranging from specialists in approximation theory and the constructive theory of functions to those working in "pure" functional analysis and geometry of convex bodies. The reader is required only to be familiar with the elements of functional analysis within the limits of a standard graduate course. The bulk of the material presented in it is due to A. L. Garkavi, R. R. Phelps and the author himself and is taken from their papers published in the decade 1956-1966. However, the ideas may often be traced as far back as to the 1858 P. L. Čebyšev treatise on the motion of certain mechanism or the 1938 N. I. Achieser and M. G. Krein book on the theory of moments, and even more direct influence of V. Klee's geometric conceptions on the development of the subject is also apparent. The deep and elegant but highly analytic theory of Hardy spaces, Mergelyan's results, methods of computation of elements of best approximation, the problem of moments and the problems related to Müntz's theorem are deliberately not included. All the harmonic analysis remains also completely outside the scope of the book. This not only limited reasonably the size of the monograph but made it possible for the author to display better the homogeneity of methods and tools and the coherence of the exposed material to the extent which was inaccessible for his predecessors. Having already described what cannot be found in the book, we are going to present now in a greater detail what kind of problems the author is interested in.

Let (E, ρ) be a metric space, $x \in E$ and $G \subset E$. A point $g_0 \in G$ is said to be an element of best approximation of x by elements of G if $\rho(g_0, x) = \inf_{g \in G} \rho(g, x)$. Denote by $\mathscr{P}_G(x)$ the set of all elements of best approximation of x by elements of G. To give, under various additional assumptions about E and G, an alternate characterization of elements of best approximation which may be more handy in a concrete situation, to say whether they do exist or are unique, to examine properties of the set-valued mapping $\mathscr{P}_G$: $E \ni x \to \mathscr{P}_G(x) \in 2^G$, and last but not least, to show a procedure to find them, these are the basic objectives for approximation theory, at least as they seem to be meant in the present book. However, in such a generality not much can be said and what can be said is briefly talked about in Appendix II (pp. 377–391). It is proved that if G is nonvoid and approximately compact (i.e. for every $x \in E$ and every $\{g_n\} \subset G$ with $\lim_{n \to \infty} \rho(x, g_n) = \rho(x, G)$ there exists a subsequence $\{g_{n_k}\}$ converging to an element of G) then it is proximinal (i.e. $\mathscr{P}_G(x)$ is nonempty for every $x \in E$) (N. V. Efimov and S. B. Stečkin, 1961) and $\mathscr{P}_G: E \to 2^G$ is upper semicontinuous (i.e. $\{x \in E | \mathscr{P}_G(x) \subset M\}$ is open for open M) (the author, 1964). We cannot either resist the temptation of recalling at this point that a closed G in a compact E is a retract of E if, and only if, there exists on E a metric ρ equivalent to the initial topology in which G is a Čebyšev set (i.e. card $\mathscr{P}_G(x) = 1$ for each $x \in E$) (C. Kuratowski, 1936, and later a generalization of W. Nitka, 1961). More can be said about the case where E is a normed linear space but G is not necessarily a linear manifold, though the theory is far from being complete. This case is dealt with in Appendix I (pp. 359–376). The Čebyševian problem of the best approximation of continuous functions by rational functions with given degrees of numerator and denominator fits well into this framework. The state of art here is best illustrated by the following facts. We do have for convex G a nice geometric characterization (*) of $g_0 \in \mathscr{P}_G(x)$ as exactly those for which there exists a real hyperplane $\mathscr{H}$ which separates G from the cell $S(x, \|x - g_0\|)$, we do know that in a reflexive and strictly convex Banach space all closed convex sets are Čebyšev sets and that this property characterizes reflexive strictly convex Banach spaces, and we do know that in a smooth Banach space E of finite dimension every Čebyšev set is convex (L. N. H. Bund, 1934, T. S. Motzkin, 1935) but, frustratingly enough, we know almost nothing about the convexity of Čebyšev sets even in a Hilbert space, let alone general normed spaces (at least for dim $\geqq 4$). The reader can also find in this Appendix some comments about the best approximation by elements of finite dimensional surfaces and arbitrary sets in which case beside the obvious statement that every proximal set is closed some more sophisticated facts about the structure of $\pi_G^{-1}(g_0)$ (R. R. Phelps, 1957–1958) are quoted without proofs.

The theory is most complete and unified in the case when E is a normed linear space and G is a linear subspace and the principal part of the text is devoted to it. The apparently more general case when G is a linear manifold can be reduced to the previous one. There are three chapters in the book, the first one (pp. 17–164) about the general case when G is an arbitrary linear subspace and remaining two discussing the best approximation by elements of finite dimensional subspaces (pp. 165–290) and from subspaces of finite codimension (pp. 291-358). Out of 61 sections of these three chapters, 25 are containing more or less direct application of the general theory in such concrete spaces as the space of real, complex or Banach-valued continuous functions on a compact set with the uniform topology, the space of real or complex continuous functions on a compact measure space endowed with L^1-norm, the L^p-spaces, $1 \leqq p \leqq \infty$, c_0, and their duals. This part being as it is undoubtedly very helpful, illustrative and containing sometimes quite nontrivial proofs is, however, a little bit outside the main stream of the material and that is why we shall not talk about it in detail.

Chapter I begins with a couple of characterizations of the elements of best approximation. One of them is a special case of (*) and is a simple corollary to the Hahn-Banach theorem and the other one in its geometric interpretation goes as follows: $g_0 \in \mathscr{P}_G(x)$ (for simplicity assume always that $x \in E\backslash\bar{G}$) if and only if there exists a real extremal hyperplane H_g (i.e. generated by a functional g that is an extreme point of the unit cell in E^*) which passes through g_0, supports the cell $S(x, \|x-g_0\|)$ and separates it from g (G. Choquet, 1963), the proof being based on the Krein-Milman theorem. There follows a paragraph on the existence of elements of best approximation or in other words about the proximinality (the term coined by R. Killgrove by a merger of "proximity" and "minimality") of G. From the corollary to the James characterization of reflexive Banach spaces we know that all closed subspaces of a Banach space E are proximinal if and only if E is reflexive. On the other hand, every normed linear space has at least one proximinal subspace (e.g. all finite dimensional subspaces). In general, it is proximinal if and only if it is closed and $G^\perp$ has the property $(\mathscr{E}_*)$ (i.e. $\forall\, x \in E$, $\exists\, y \in E$ such that $\gamma(y) = \gamma(x)$, $\gamma \in G^\perp$ and $\|y\| = \|x\|_{G^\perp}$). Other characterizations are provided too. Next the uniqueness problems are studied. G is said to be semi-Čebyšev if card $\mathscr{P}_G(x) \leqq 1$, $\forall\, x \in E$, so that obviously it is Čebyšev when it is both semi-Čebyšev and proximinal. By adding to the condition $(\mathscr{E}_*)$ a requirement to the effect that the existing $y \in E$ be unique, and calling this new condition $(\mathscr{U}_*)$ the author gets a quick characterization of semi-Čebyšev subspaces G as those for which $G^{\perp}$ has the property $(\mathscr{U}_*)$. All subspaces of E are semi-Čebyšev if and only if E is strictly convex (K. Tatarkiewicz, 1952) and in every normed space there is at least one semi-Čebyšev subspace. In a Banach space one can even always find a semi-Čebyšev hyperplane (J. Lindenstrauss, 1966), but on the other hand there are Banach spaces with no Čebyšev subspace although it is not yet known whether such an example can be found among separable Banach spaces. At the end of the first chapter the multivalued, in general, mapping π_G: $\mathrm{Dom}(\pi_G) \to G$ defined by the condition $\pi_G(x) \in \mathscr{P}_G(x)$ and the functional $e_G(x) = \rho(x, G)$ is studied. Neither π_G nor e_G are in general linear should G be even a Čebyšev subspace. However, π_G is homogeneous, idem-potent and continuous at the origin and it admits an additive selection if G is a proximinal hyperplane passing through 0 thus warranting the linearity in the case when G is a Čebyšev hyperplane with $0 \in G$ (N. Aronszajn and K. T. Smith, 1954, R. A. Hirschfeld, 1958). $\pi_G^{-1}(g_0)$, $g_0 \in G$, is always closed (R. R. Phelps, 1958). e_G is subadditive, positively homogeneous and continuous to mention only a few of its properties, and its properties for increasing or decreasing sequences $\{G_n\}$ of closed linear subspaces are also described in this part following the work of V. N. Nikolsky and I. S. Tyuremskih.

The lines of the next two chapters follow essentially the schedule of the first one taking advantage of finite dimension or codimension of G and making extensions and specializations of previous results where it is possible. So as a specialization of the first characterization of the elements of best approximation (which, in the case $\dim G = n < \infty$ are reasonably called polynomials of best approximation) from Chapter I we get that $g_0 \in \mathscr{P}_G(x)$, $\dim G = n < \infty$, if and only if there exists h extreme points $f_1, \ldots, f_h$ of the unit cell S_{E^*} where $1 \leqq h \leqq n+1$ if the scalars are real and $1 \leqq h \leqq 2n+1$ if the scalars are complex and h numbers $\lambda_1, \ldots,$ $\lambda_h > 0$ with $\sum \lambda_j = 1$ such that

$$\sum_{j=1}^{h} \lambda_j f_j(g) = 0 \quad (g \in G), \quad \text{and} \quad \sum_{j=1}^{h} \lambda_j f_j(x - g_0) = \|x - g_0\|$$

(the author, 1965). The arguments in Chapter III are usually dual to those of Chapter II and a part of the results is formulated in dual spaces with weak* topology. The only novelty here is sections on n-dimensional diameters (respectively on diameters of order n in Chapter III) and best

n-dimensional secants. If A is a subset of E then the number

$$d_n(A) = \inf_{\dim G = n} \sup_{x \in A} e_G(x)$$

is called the n-dimensional diameter of A and an n-dimensional subspace $G \subset E$ for which $\sup_{x \in A} e_G(x) = d_n(A)$ is called the best n-dimensional secant of A. Both notions are due to A. N. Kolmogorov (1936). The first one may be viewed as a sort of measure of the n-dimensional "thickness" of A and the rate of decay of $d_n(A)$, $n \to \infty$, may be considered as a natural measure of the size of A. The interesting connections that exist between diameters and another similar measure, namely the ε-entropy, are regrettably not studied here. The fundamental result of these sections says that if E_{n+1} is an $(n + 1)$-dimensional subspace of E then $d_n(S_E \cap E_{n+1}) = 1$ (V. M. Tihomirov, 1960) and the problem of existence of best n-dimensional secants is another main topic here.

What we have written above did not strive to be a complete presentation of contents of the monograph under review, but was rather aimed at giving a reader a taste of what was there without mentioning even the vast area of applications which possess flavors of their own. The bibliography is fairly complete and only the lack of the entry for memoir [3] in it with which Singer's book shares a number of topics raises our objections. The author must be given a credit for tracing the history of all results very carefully and in great detail. Sometimes, in deciding about the priorities he goes as far as calling to the witness stand other papers as in the following sentence: "Theorem 11 has been given and used for the first time in the paper [...], this fact is explicitly mentioned, e.g., in the paper of R. C. Buck [...], p...." (p. 22). In comparison to other books on the subject we have gotten the impression that this monograph displays more formalistic approach (or more modern as some would say) with very little attention being paid to the heuristic background. The author is generous in giving spare proofs (e.g., Theorems 1.5, 1.6, Chapter I) and gives in some characterization theorems literally dozens of equivalent conditions that, we must concede, do not make the text more transparent but undoubtedly will make the life easier for people who will be willing to cite results that fit exactly their needs. N. I. Achieser's classic [4] and G. Lorentz' [5] contemporary of Singer's original are both much less general and much less detailed in the presentation of the material which is dealt with in the book but, instead, they offer the chapters on harmonic analysis with the whole Paley-Wiener theory [4], on entropy, approximative dimension and the Kolmogorov's theorem on representation of functions of several variables by functions of one variable [5]. The brand-new lecture notes by H. Shapiro [6] are again less general and stick to the problems of uniform approximation, L^p-norms and Hilbert spaces stressing in the latter case the usefulness of reproducing kernels. Still they treat a more wide range of topics including Müntz' theorem, minimal extrapolation of Fourier transform and applications in complex analysis and feature in the introduction a good deal of the author's "philosophy" of approximation theory.

To be fair to the reader we have to admit that the book under review presents the state of art as of 1966 and that the present English edition was not updated. As of this writing, however, there appeared a great many papers having made a substantial contribution to the approximation theory in normed spaces. Obviously it is not the job to be done in this review to analyze the recent development of the subject but for the reader's convenience we attach to the references a short list of papers [7]–[10] which, as we feel, represent some of the recent trends and extend in some way or another, results presented by the useful book of Ivan Singer.

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According to the foreword, *Algebraic Number Theory* is meant to supersede Lang's *Algebraic Numbers.* The earlier book could be split roughly into two parts: the first half was basic algebraic number theory,