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UNIFORM ESTIMATES FOR THE $\bar{\partial}$-EQUATION ON INTERSECTIONS OF STRICTLY PSEUDOCONVEX DOMAINS[1]

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1. A few years ago Grauert-Lieb [5] and Henkin [3] solved independently the $\bar{\partial}$-problem with uniform bounds for (0, 1)-forms on strictly pseudoconvex domains in C^n with smooth boundary, i.e., they proved that for every bounded $\bar{\partial}$-closed $C^{\infty}_{(0,1)}$-form f on such a domain D there is a bounded C^{∞} function u on D with $\bar{\partial}u = f$. In the proofs, the solution u is constructed explicitly in terms of integrals involving a holomorphic kernel constructed by Henkin [2] and Ramirez [10]. Kerzman [7] extended this idea to obtain a local version of the above result, which enables him to get the same result for strictly pseudoconvex domains with a smooth boundary in a Stein manifold. Based on results of Koppelman [8], Lieb [9] then obtained uniform estimates for the $\bar{\partial}$-equation for $(0, q)$-forms on the same class of domains. Recently Henkin [4] announced the solution of the $\bar{\partial}$-problem with uniform bounds for (0, 1)-forms on certain analytic polyhedra.

In this note we announce the solution of the $\bar{\partial}$-problem with uniform bounds for $(0, q)$-forms on a domain which is the intersection of a finite number of strictly pseudoconvex domains intersecting normally. As a consequence of this result, for any complex manifold, one can find a locally finite open covering such that the $\bar{\partial}$-equation can be solved with uniform bounds on intersections of members of the covering. This essentially answers a question raised by Gunning [6, p. 73].

2. To state our result precisely, let D be a bounded domain in C^n such that there exist a finite open covering $\{U_j\}_{j=1}^k$ of ∂D and strictly plurisubharmonic C^4 functions $\rho_j: U_j \to R$, $1 \leqq j \leqq k$, such that $D \cap (\bigcup_{j=1}^k U_j)$ is the set of all $x \in \bigcup_{j=1}^k U_j$ which, for every $1 \leqq j \leqq k$, satisfy $x \notin U_j$ or $\rho_j(x) < 0$. Set $S_j = \{x \in U_j \cap \partial D : \rho_j(x) = 0\}$. We assume that for every sequence $1 \leqq i_1 < \ldots < i_l \leqq k$, $1 \leqq l \leqq 2n$, and for every $x \in \bigcap_{v=1}^l S_{i_v}$ the 1-forms $d\rho_{i_1}, \ldots, d\rho_{i_l}$ are linearly independent over R at x.

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For a form $f \in C^{\infty}_{(0,q)}(D)$, $1 \leqq q \leqq n$, the uniform norm $\|f\|_{\infty}$ is defined as the maximum of the supremum norms over D of all the coefficients of f.

THEOREM 1. *Let D be a domain satisfying the conditions stated above. Then there is a constant $K < \infty$ such that, for any $f \in C^{\infty}_{(0,q)}(D)$ $(1 \leqq q \leqq n)$ with $\bar{\partial} f = 0$ and $\|f\|_{\infty} < \infty$, there exists $u \in C^{\infty}_{(0,q-1)}(D)$ such that*
(i) $\bar{\partial} u = f$ on D,
(ii) $\|u\|_{\infty} \leqq K \|f\|_{\infty}$.
Moreover, u extends continuously to the boundary of D.

The proof of Theorem 1, besides using a rather straightforward combination of Henkin's method for the analytic polyhedron case [4] and Lieb's method [9], depends essentially on some nontrivial estimations of integrals. The estimates are actually good enough to obtain a uniformly bounded solution u even if f is only bounded in L^p norm for sufficiently large p.

The constant K can be chosen in such a way that Theorem 1 holds with the same K for small perturbations of D, for example for the domains

$$D_{\varepsilon} = \left(D - \bigcup_{j=1}^{k} U_j\right) \cup \left\{x \in \bigcup_{j=1}^{k} U_j : x \notin U_j \text{ or } \rho_j(x) < \varepsilon, 1 \leqq j \leqq k\right\}$$

whenever $|\varepsilon|$ is sufficiently small.

3. Theorem 1 implies the vanishing of the cohomology groups $H^q(\bar{D}, \mathfrak{B})$ for $q \geqq 1$, where $\mathfrak{B}$ is the sheaf of germs of bounded holomorphic functions on $\bar{D}$ (see [9], where this is proved for strictly pseudoconvex domains with smooth boundary). Thus, for $q \geqq 1$, the bounded cohomology agrees with the ordinary cohomology. For a more restricted class of domains we also obtain the corresponding result for the multiplicative cohomology.

More precisely, let D be a domain as defined in §2. Moreover, we now assume that $k < n$ and that for all sequences $1 \leqq i_1 < \ldots < i_l \leqq k$, $1 \leqq l \leqq k$, and all $x \in \bigcap_{\nu=1}^{l} S_{i_\nu}$ the (1, 0)-forms $\partial\rho_{i_1}, \ldots, \partial\rho_{i_l}$ are linearly independent over C at x. Let $\mathfrak{O}^*$ be the multiplicative sheaf of germs of nowhere zero holomorphic functions, and let $\mathfrak{B}^*$ be the sheaf on $\bar{D}$ associated to the presheaf $W \mapsto \mathfrak{B}^*(W)$, where $\mathfrak{B}^*(W)$ is the multiplicative group of all $f \in \Gamma(W \cap D, \mathfrak{O}^*)$ with f and f^{-1} bounded on $W \cap D$.

THEOREM 2. *Under the hypotheses on D stated above the natural homomorphism*

$$H^q(\bar{D}, \mathfrak{B}^*) \to H^q(D, \mathfrak{O}^*)$$

is an isomorphism for $q \geqq 1$.

The special case of Theorem 2 where $q = 1$ and D is a (Euclidean) strictly convex domain with smooth boundary was proved by Stout [11]. The proof of Theorem 2 uses a modification of Stout's method combined with techniques due to Bishop [1].

MAIN IDEA OF THE PROOF. By taking logarithms, the bounded multiplicative problem is reduced to an additive problem for holomorphic functions with bounded real parts. The main difficulty now lies in estimating the imaginary part. Such estimates exist for L^p norms ($p < \infty$) on the unit disc in C. In order to extend them to D, we use Bishop's techniques to parametrize the real submanifolds of C^n making up the boundary of D by suitable families of boundaries of analytic discs. (It is in this step that one needs the additional restrictions on D.) Thus the differential forms obtained by solving the additive problem are bounded in L^p norm for large p. As remarked after Theorem 1, this is sufficient to obtain uniformly bounded solutions for the $\bar{\partial}$-equation.

Full details and generalizations will appear elsewhere.

ADDED IN PROOF. We have learned from G. M. Henkin that P. L. Poljakov (Uspehi Mat. Nauk 26 (1971), 243–244) has announced a result similar to Theorem 1.

REFERENCES

1. E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21. MR #369.

2. G. M. Henkin, *Integral representations of functions holomorphic in strictly pseudoconvex domains and some applications*, Mat. Sb. **78** (**120**) (1969), 611–632 = Math. USSR Sb. 7 (1969), 597–616. MR **40** #2902.

3. ———, *Integral representation of functions in strictly pseudoconvex domains and applications to the $\bar{\partial}$-problem*, Mat. Sb. **82** (**124**) (1970), 300–308 = Math. USSR Sb. **11** (1970), 273–281. MR **42** #534.

4. ———, *Uniform estimates for solutions to the $\bar{\partial}$-problem in Weil domains*, Uspehi Mat. Nauk **26** (1971), 211–212 (Russian).

5. H. Grauert and I. Lieb, *Das Ramirezsche Integral und die Lösung der Gleichung $\bar{\partial}f = \alpha$ im Bereich der beschränkten Formen*, Complex Analysis (Proc. Conf. Rice Univ., Houston, Tex., 1969), Rice University Studies, vol. 56, no. 2, Rice University, Houston, Tex., 1970, pp. 29–50. MR **42** #7938.

6. R. C. Gunning, *Local moduli for complex analytic vector bundles*, Math. Ann. **195** (1971), 51–78.

7. N. Kerzman, *Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains*, Comm. Pure Appl. Math. **24** (1971), 301–379.

8. W. Koppelman, *The Cauchy integral for differential forms*, Bull. Amer. Math. Soc. **73** (1967), 554–556. MR **35** #6862.

9. I. Lieb, *Die Cauchy-Riemannschen Differentialgleichungen auf streng pseudokonvexen Gebieten*, Math. Ann. **190** (1970), 6–44.

10. E. Ramirez de Arellano, *Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis*, Math. Ann. **184** (1969/70), 172–187. MR **42** #4767.

11. E. L. Stout, *On the multiplicative Cousin problem with bounded data*, preprint.

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