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ON THE EXISTENCE OF A "WAVE OPERATOR" FOR THE BOLTZMANN EQUATION[1]

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ABSTRACT. The Boltzmann equation is considered on the appropriate Hilbert space. The nonlinear problem is looked at as a perturbation of its linearized version. Thus, one deals with a pair of contractive semigroups, and a "wave operator" for this pair is studied. We find a subspace of finite codimension where the corresponding limit exists.

The Boltzmann equation for a monoatomic gas is

$$\begin{aligned} \partial f/\partial t + v_1 \cdot \operatorname{grad} f &= Bf \\ &= \iint [f(v_2^*)f(v_1^*) - f(v_2)f(v_1)] \\ &\qquad \cdot |v_1 - v_2| I(|v_1 - v_2|, \theta) \sin\theta \, d\theta \, d\phi \, dv_2. \end{aligned} \tag{1}$$

Here $f(t, r, v)$ is the velocity distribution function at time t at the point r, and the star on v_1 and v_2 denotes the effect of a binary collision. $I(|v_1 - v_2|,\theta)$ is the differential scattering cross section corresponding to the turning of the relative velocity $v_1 - v_2$ in an interaction.

We are concerned with the spatially homogeneous case and moreover we assume that we are dealing with a cut-off interaction, so that

$$\int I(v, \theta) \sin\theta \, d\theta \, d\phi < \infty. \tag{2}$$

Under these restrictions the initial value problem for the Boltzmann equation has been much studied.

There is one molelular interaction, proposed by Maxwell, which simplifies the mathematics in (1) a bit. One proposes a central potential inversely proportional to r^4 and one finds that $vI(v, \theta)$ is a function of θ alone, with a pole at $\theta = 0$. This pole is removed by the cut-off assumption (2). Thus the equation can be written as

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(3) $$\partial f/\partial t = f * f - f$$

with

(4) $$(f * f)(v_1) = \iint f(v_2^*)f(v_1^*)|v_1 - v_2|I(|v_1 - v_2|, \theta)\sin\theta \, d\theta \, d\phi \, dv_2.$$

In (3) we are taking the total cross section to be unity.

Define $g(v) = (2\pi)^{-3/2} \exp - v^2/2$. Then (3) can be considered as an initial value problem on the submanifold of $L^2(g^{-1})$ given by those functions which are positive and satisfy the five scalar conditions

(5) $$\int f(v)\, dv = 1, \quad \int f(v)v \, dv = 0, \quad \int f(v)v^2 \, dv = 1.$$

It turns out that (3) is well posed in a sufficiently small ball centered around g. This result is well known and not particularly hard to prove. One also proves that g is an attractive center for the flow given by (3), so that any initial datum in the vicinity of g approaches it as time increases.

Our aim is to compare the actual flow (3), subject to the conditions (5), with its linearized version around g. One writes $f = g + h$, notices that $g * g = g$ and then drops from

$$\dot{h} = f * f - f = (g * h + h * g - h) + h * h$$

the nonlinear term in h, obtaining

(6) $$\dot{h} = Ah = g * h + h * g - h.$$

The treatment of (6) is greatly simplified by the fact that A is a negative selfadjoint operator having a purely discrete spectrum. See [**1**] and [**2**]. Let Q_t and T_t denote the semigroups relating data at time 0 to its evolution at time t for equations (3) and (6) respectively.[2]

We are interested in proving the existence of a nonlinear change of coordinates, around g, that would convert the nonlinear problem (3) into the linear one given by (6). Explicitly, we want to find a very smooth mapping ψ from a neighborhood of g into itself, that leaves g invariant, coincides with the identity up to first order, and satisfies

(7) $$Q_t = \psi^{-1} T_t \psi$$

for all positive times.

One shows easily that such a ψ is readily available if

(8) $$\lim_{t \to +\infty} T_{-t} Q_t$$

[2] Actually we want $T_t(g + h) = g + h_t$ where h_t is the evolution of h according to (6).

exists and is invertible close to g. This is a common procedure in scattering theory where one deals with two unitary groups. The author found that the trick works for the Boltzmann equation too. Here one is dealing with contractive semigroups and one of them is nonlinear. The main differences with the unitary case are that (a) even the finite-dimensional case is interesting, (b) limits can exist only in one sense and not as $t \to \pm\infty$.

Limits like that in (8) are called wave operators in scattering theory. A mapping like the ψ in (7) was first considered by Poincaré. See [3] and its references.

In [3] a complete study of a simpler model of (3), introduced by Kac [4], is done and the existence of the limit (8) is established.

In the 3-dimensional case the situation is much more involved and the result is changed a bit. Although a ψ satisfying (7) can still be found, the limit in (8) does not exist for a general initial data. Remarkably enough one can exhibit a large subspace where the limit does exist. It turns out to be a subspace of codimension 3, in the appropriate Hilbert space, given in a rather simple way. This is the content of the following:

THEOREM. *If $f \in L^2(g^{-1})$, is close enough to g, and satisfies not only* (5) *but also the three extra scalar requirements*

$$\int f(v)vv^2\, dv = 0, \tag{9}$$

i.e. the "heat flow vector" vanishes, then

$$\lim_{t\to+\infty} T_{-t}Q_t f \quad \textit{exists.}$$

Condition (9) is both sufficient and necessary for the existence of the limit. The necessity could already have been established by Maxwell himself. Indeed, using the notation in [1] or [2], the eigenvalues of A are

$$\lambda_{rl} = 2\pi \int_0^\pi \sin\theta\, F(\theta)\, d\theta \left[\cos^{2r+l}\frac{\theta}{2} P_l\left(\cos\frac{\theta}{2}\right) + \sin^{2r+l}\frac{\theta}{2} P_l\left(\sin\frac{\theta}{2}\right) - (1+\delta_{0l}\delta_{0r})\right].$$

Set $A_{2k} = 2\pi \int_0^\pi (\sin\theta)^{2k+1}F(\theta)\, d\theta$ and conclude that

$$\lambda_{11} = -\tfrac{1}{2}A_2, \qquad \lambda_{22} = -\tfrac{9}{8}A_2 + \tfrac{3}{16}A_4.$$

Now condition (9) would be unnecessary only if one could prove the inequality $2\lambda_{11} < \lambda_{22}$. This is equivalent to $2A_2 < 3A_4$. Maxwell [5] had already computed A_2 and A_4 with such an accuracy that he could have ruled out the inequality above.

The sufficiency of (9) is, of course, harder to establish.

We cannot, unfortunately, give any physical explanation for the result above. It is not even clear that there should be any. Instead, the proof is based on a careful study of the spectral properties of the operator A in (6). The proof depends heavily on the use of the Talmi transformation [6] and the numerical computation of a large number of eigenvalues of A done by Alterman, Frankowski and Pekeris [7].

The transformation referred to above was introduced by Talmi in a study of the harmonic oscillator shell model of nuclear physics. The connection hinges on the fact that the eigenfunctions of A are those of the harmonic oscillator. Kumar [8] introduced the Talmi transformation in kinetic theory.

The numerical computation of the eigenvalues of A – only the first 559! – turns out to be very useful to supplement analytical facts in proving some crucial "eigenvalue inequalities". See [3] and [9].

A proof of the theorem as well as related results will be published elsewhere.

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