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A NOTE ON POINCARÉ 2-COMPLEXES

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The purpose of this note is to announce some progress on the following conjecture:

CONJECTURE. Every Poincaré 2-complex is of the homotopy type of a closed 2-manifold.

By connected Poincaré n-complex we mean a connected CW complex X dominated by a finite CW complex which satisfies Poincaré duality with local coefficients: Let $\pi = \pi_1 X$ and let $\Lambda = Z\pi$ be the group ring of π. Let $\omega : \pi \to \{\pm 1\}$ be a homomorphism (trivial if X is to be "oriented"). Let $\bar{\Lambda}$ be the right π-module whose elements are the same as Λ but the right action is given as follows: For $\lambda \in \bar{\Lambda}$, $x \in \pi$, $\lambda \cdot x = \omega(x)x^{-1}\lambda$. Then there exists some class $[X] \in H_n(X; Z \otimes_\Lambda \bar{\Lambda})$ such that $[X] \cap : H^i(X; \Lambda) \to H_{n-i}(X; \bar{\Lambda})$ is an isomorphism for all i.

Wall's results [1] give the following ($\simeq$ means "homotopy equivalent to"):

THEOREM (WALL). *Let X be a connected Poincaré 2-complex. Let $\pi = \pi_1 X$. Then*

(a) *if π is finite, $X \simeq S^2$ or RP^2;*

(b) *if π is infinite then X is a $K(\pi, 1)$;*

(c) *there exists a unique 2-manifold M_X such that $H_*(X; Z) \simeq H_*(M_X; Z)$ (simple coefficients);*

(d) *$X \simeq X'$ a CW complex of dimension $\leqq 3$.*

Thus the conjecture becomes, more specifically: If X is a Poincaré 2-complex, then $X \simeq M_X$. The results we have obtained so far are the following:

THEOREM. *Let X be a connected finite Poincaré 2-complex; then*

(a) *if $M_X = S^2$ or RP^2 then $X \simeq M_X$,*

(b) *if X is 2-dimensional as a CW complex and $M_X = S^1 \times S^1$ or the Klein bottle, then $X \simeq M_X$.*

In both (a) and (b) the unoriented case follows from the oriented: If $M_X = RP^2$ (resp. the Klein bottle), then it can be shown that X', a certain double cover of X, is a Poincaré 2-complex with $M_{X'} = S^2$ (resp. $S^1 \times S^1$). Assuming the oriented case, we get $X' \simeq S^2$ (resp. $S^1 \times S^1$).

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It then follows (in case (a), easily from a result of Wall; in case (b), with some amount of algebraic manipulation) that $X \simeq RP^2$ (resp. the Klein bottle).

For the case $M_X = S^2$, we may as well assume $X = K(\pi, 1)$ (since the finite case is solved by Wall). Then there is a free finite π-resolution of the trivial π-module Z:

$$(*) \qquad 0 \to G_2 \xrightarrow{\alpha} G_1 \xrightarrow{\beta} \Lambda \to 0.$$

By using the assumptions on X we get that if $n = \text{rank } G_2$, $n - 1 = \text{rank } G_1$. By Poincaré duality, $(**) = \text{Hom}_\Lambda(*, \Lambda)$:

$$(**) \qquad 0 \leftarrow G_2^* \xleftarrow{\alpha^*} G_1^* \xleftarrow{\beta^*} \Lambda \leftarrow 0$$

has homology Z in dimension 2, 0 elsewhere. We can choose generators so that $G_2^* = M \oplus N$ where M has rank $(n - 1)$ and $M \subset \text{im } \alpha^*$ and $N \simeq \Lambda$. Let $\pi : G_2^* \to M$ be the projection. Then $\pi\alpha^* : G_1^* \to M$ is an epimorphism of free Λ-modules of rank $(n - 1)$. But a theorem of Kaplansky (unpublished) states that if R is an integral (or complex) group ring then an epimorphism of free modules of the same finite rank is an isomorphism. Thus $\pi\alpha^*$ is an isomorphism so α^* is a monomorphism and thus $\beta^* = 0$, clearly a contradiction since β^* is a monomorphism from a nontrivial module.

The case $M_X = S^1 \times S^1$ is more difficult. Here $\pi = \langle a_1, \ldots, a_{n+1} \mid \alpha_1, \ldots, \alpha_n \rangle$ (since $X = K(\pi, 1)$ is a 2-dimensional CW complex). The abelianization of π, $\pi^{\text{ab}} \simeq Z \oplus Z$ and we can assume that the map $\pi \to \pi^{\text{ab}}$ sends $a_i \to 0$, $i < n$, and $a_n \to (1,0)$, $a_{n+1} \to (0,1)$. Let $\pi' = [\pi,\pi]$, F be the free group on $a_1, \ldots, a_n$, N the smallest normal subgroup of F containing $\alpha_1, \ldots, \alpha_n$, and K the smallest normal subgroup of F containing $a_1, \ldots, a_{n-1}, [a_n, a_{n+1}]$. Then there is an epimorphism $K \to \pi'$ with kernel N, i.e. $1 \to N \xrightarrow{i} K \to \pi' \to 1$ is exact. Thus there is an exact sequence

$$N^{\text{ab}} \xrightarrow{i^{\text{ab}}} K^{\text{ab}} \to \pi'^{\text{ab}} \to 0.$$

We wish to show two things:

(1) π' is free and

(2) i^{ab} is an epimorphism.

If these are proved then π' must be trivial so that $\pi \simeq Z \oplus Z$ so $X \simeq S^1 \times S^1$.

(1) is not difficult. It follows from the following:

LEMMA. *Let X be a Poincaré 2-complex and let X' be a covering space corresponding to a subgroup of $\pi_1 X$ of infinite index. Then $X' \simeq$ a wedge of circles.*

PROOF. A straightforward proof shows that if $\pi' = \pi_1 X'$, then $H^i(X';A) \simeq H^i(X; Z\pi \otimes_{Z\pi'} A) \simeq H_{2-i}(X; Z\pi \otimes_{Z\pi'} A)$ (with a twist for the unoriented case). This latter group is obviously 0 for $i > 2$ and since $[\pi:\pi'] = \infty$, it is 0 for $i = 2$. Thus since $X' = K(\pi',1)$ (because $X = K(\pi,1)$), π' is a group of cohomological dimension 1, hence by the Stallings-Swan Theorem, is free. □

Proving (2) is much more difficult. Here, one has to use the fact that N^{ab} and K^{ab} are $Z\pi$-modules on generators $\{\alpha_1, \ldots, \alpha_n\}$ and $\{a_1, \ldots, a_{n-1}, [a_n, a_{n+1}]\}$ respectively. K^{ab} is also a $Z\pi^{ab}$-module on the same generators. We then investigate $i^{ab}\{\alpha_1, \ldots, \alpha_n\}$ and show that the matrix it represents (with coefficients in $Z\pi^{ab}$) in terms of $\{a_1, \ldots, a_{n-1}, [a_n, a_{n+1}]\}$ is invertible if an only if $H^2(X;\Lambda) \simeq Z$ which we know since $H^2(X;\Lambda) \simeq H_0(X;\Lambda)$. The theory of derivations is used extensively: A derivation is a function $d: G \to M$ where G is a group and M a left G-module, satisfying $d(xy) = dx + x\,dy$. It turns out that both $H^2(X;\Lambda)$ and $i^{ab}\{\alpha_1, \ldots, \alpha_n\}$ are expressible in terms of certain derivations.

The details of these theorems will appear shortly. The cases $M_X = S^2$ and RP^2 will appear in [2].

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