

THE ENTIRE CHROMATIC NUMBER OF A NORMAL GRAPH IS AT MOST SEVEN

BY HUDSON V. KRONK¹ AND JOHN MITCHEM²

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A multigraph is said to be *normal* if it is embedded in the plane such that each vertex is adjacent to exactly three edges and three regions. In [2], G. Ringel showed that the vertices and regions of a normal multigraph can be colored with six colors such that adjacent elements are colored differently. In this note we consider the problem of coloring vertices, regions, and edges of normal multigraphs. Formally, the *entire chromatic number* of a plane multigraph G is the fewest number of colors required to color the vertices, regions, and edges of G so that adjacent elements are colored differently. Here a region is adjacent to the vertices and edges which are on its boundary. Also a vertex is adjacent to its incident edges. In [1], H. Izbicki reported that by assuming the four color conjecture M. Neuberger has proved

THEOREM. *The entire chromatic number of any normal multigraph is at most seven.*

The theorem is best possible in the sense that the complete graph on four vertices requires seven colors. We prove the theorem without using the four color conjecture.

OUTLINE OF PROOF. We use induction on r , the number of regions of normal multigraph G . It is easily verified that $r \geq 3$, and if $r = 3$, the Theorem holds. Four different cases are then considered depending on the region R of G which is adjacent to the fewest edges of G . In each subcase, region R is removed from G in such a way that the resulting multigraph H is normal. By the inductive assumption, H has a seven coloring. From this, a seven coloring of G is developed. In the case where R is adjacent to five edges of G , as well as the case where R is adjacent to four edges of G , it is necessary to consider a number of subcases depending on the coloring of H .

We also conjecture that

CONJECTURE. The entire chromatic number of any plane multigraph is at most four more than its maximum degree.

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REFERENCES

1. H. Izbicki, "Verallgemeinerte Farbenzahlen," *Beiträge zur Graphentheorie*, H. Sachs, H. Voss and H. Walther (Editors), Teubner, Leipzig, 1968, pp. 81–84.
2. G. Ringel, *Ein Sechsfarbenproblem auf der Kugel*, *Abh. Math. Sem. Univ. Hamburg* **29** (1965), 107–117. MR **32** # 4685.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON,
BINGHAMTON, NEW YORK 13901

DEPARTMENT OF MATHEMATICS, SAN JOSE STATE COLLEGE, SAN JOSE, CALIFORNIA 95204