The purpose of this note is to give an elementary proof of the following result.

**Theorem A.** Let $G$ be a finitely generated nonelementary Kleinian group and let $J$ be an anticonformal homeomorphism of $\Omega = \Omega(G)$, the set of discontinuity of $G$, where $J$ commutes with every element of $G$. Then $J$ is the restriction of an anticonformal, involutory fractional linear transformation (that is, $J(z) = \frac{az + b}{cz + d}$, $J^2 = 1$) and $G$ is either Fuchsian or a $\mathbb{Z}_2$-extension of a Fuchsian group. Further, the mapping $J$ with the above properties is unique.

We prove Theorem A by reducing it to

**Theorem B.** Let $\Gamma$ be a finitely generated Fuchsian group operating on $U_1$ and $U_2$, the upper and lower half-planes, respectively. Let $f_1$ and $f_2$ be schlicht functions on $U_1$ and $U_2$, where $f_1 \circ \gamma \circ f_1^{-1}$ and $f_2 \circ \gamma \circ f_2^{-1}$ both define the same isomorphism of $\Gamma$ onto a Kleinian group $G$, and $f_1 = f_2$ on that part of the real axis $R$ lying in $\Omega(\Gamma)$. Then $f_1$ and $f_2$ are restrictions of the same fractional linear transformation.

As a corollary to our proof of Theorem B, we obtain the somewhat more general

**Theorem C.** Let $\Gamma$ be a finitely generated Fuchsian group of the first kind acting on $U_1$ and $U_2$. Let $f_1$ defined on $U_1$, and $f_2$ defined on $U_2$ be holomorphic cover mappings where $f_1 \circ \gamma \circ f_1^{-1}$ and $f_2 \circ \gamma \circ f_2^{-1}$ both define the same homomorphism of $\Gamma$ onto a Kleinian group $G$. Then $G$ is either Fuchsian or a $\mathbb{Z}_2$-extension of a Fuchsian group (perhaps of the second kind).

**Remark.** Theorem C gives information about certain deformations of $\Gamma$, in the sense of Kra [6], where the same deformation is supported in both $U_1$ and $U_2$. Nothing is known about the more general case where $f_1$ and $f_2$ are merely locally schlicht.
Using standard techniques in quasiconformal mappings, we also get an elementary proof of the following result of Maskit [9].

**Theorem D.** Let $G$ be a finitely-generated Kleinian group with two invariant components. Then $G$ is a quasiconformal deformation of a Fuchsian group.

We start by giving a proof of Theorem B. We denote the map $z \mapsto \bar{z}$ by $j$. Note that we have a well defined mapping $f: \Omega(\Gamma) \to \Omega(G)$; *a priori* this mapping need not be surjective. It projects to a surjective mapping-display $f^*: \Omega(\Gamma)/T \to f(\Omega(\Gamma))/G$. Since $\Omega(\Gamma)/T$ is of finite type, $f(\Omega(\Gamma))$ is a union of components of $\Omega(G)$ and $f^*$ is an $n$-sheeted covering for some $n \geq 1$. Furthermore, $f^*|(U_i/\Gamma)$ is injective for $i = 1, 2$. Thus $n = 1$ or $n = 2$.

If $f_1(U_1) \cap f_2(U_2) = \emptyset$, then $G$ has two invariant connected open subsets of its region of discontinuity: namely $f_1(U_1)$ and $f_2(U_2)$. Thus every noninvariant component of $\Omega(G)$ is an atom (Accola [1]). Since finitely generated Kleinian groups do not have atoms (Ahlfors [2]) we conclude that

$$\Omega(G) = f_1(U_1) \cup f_2(U_2) \cup f_1(\Omega(\Gamma) \cap R).$$

Thus if $f_1(U_1) \cap f_2(U_2) = \emptyset$, we set

$$J(z) = f_2 \circ j \circ f_1^{-1}(z), \quad z \in f_1(U_1),$$

$$= f_1 \circ j \circ f_2^{-1}(z), \quad z \in f_2(U_2),$$

$$= z, \quad z \notin f_1(U_1) \cup f_2(U_2),$$

and observe that $J^{-1} \circ g \circ J = g$ for every $g \in G$.

Since $f_1$ and $f_2$ both are equivalent (under the Möbius group) to bounded holomorphic functions, by Fatou’s Theorem they have locally $L_1$ (even $L_\infty$) vertical boundary values. Using the Cauchy integral formula it suffices to show that these are the same a.e. Observe that by Maskit [10], $J$ is a homeomorphism. Now if $f_1(w)$ has a limit as $\text{Im } w \to 0$, then either $w$ approaches a point in $\Omega(\Gamma)$, in which case by hypothesis, $f_2(jw)$ tends to the same point as $f_1(w)$; or, since $w$ is schlicht, $f_1(w)$ tends to a point of $\Lambda(G)$, the limit set of $G$. If $f_1(w)$ tends to a point of $\Lambda(G)$, then $f_2(jw) = J(f_1(w))$ tends to the same point.

If $f_1(U_1) \cap f_2(U_2) \neq \emptyset$, then observe that $f_1(U_1)$ is bounded by the limit points of $G$ and the points in the image of $\Omega(\Gamma) \cap R$, and so $f_1(U_1) = f_2(U_2)$. Then $f_2^{-1} \circ f_1$ is directly conformal, maps $U_1$ onto $U_2$, and is the identity on $\Omega(\Gamma) \cap R$ and on the hyperbolic fixed points. Hence, $f_1(U_1) \cap f_2(U_2) \neq \emptyset$ cannot occur.
REMARK. A more direct proof of Theorem B can be obtained by showing that $|f_1(z) - f_2(jz)|$ tends uniformly to zero as $\text{Im} \, z \to 0$ with $z \in U_1$. This involves an analysis similar to the one appearing in Maskit [10].

PROOF OF THEOREM A. We first observe that $J^2$ is conformal and commutes with every element of $G$. Hence (Kra [7] or Maskit [10]), $J^2 = 1$.

Suppose there is a component $\Delta_1$ of $G$ with $\Delta_2 = J\Delta_1 \neq \Delta_1$. Let $H$ be the subgroup of $G$ keeping $\Delta_1$ invariant; obviously $H\Delta_2 = \Delta_2$. By Accola's remark [1], $\Delta_1$ and $\Delta_2$ are both simply-connected. Choose a Fuchsian group $\Gamma$ and a conformal map $f_1 : U_1 \to \Delta_1$ which conjugates $\Gamma$ into $G$. Define $f_2 : U_2 \to \Delta_2$ by $f_2 = J \circ f_1 \circ j$. Since $\Gamma$ is of the first kind, by Theorem B, $f_1$ and $f_2$ are restrictions of the same fractional linear transformation $f$. Then in $\Delta_2$, $J = f \circ j \circ f^{-1}$; and so $J = f \circ j \circ f^{-1}$ everywhere.

We now assume that $J$ keeps every component of $G$ invariant. Then $G$ has only one component $\Delta$, for set

$$J^*(z) = Jz, \quad z \in \Delta,$$

and observe that, by Maskit [10], $J^*$ is a global homeomorphism which reverses orientation in $\Delta$, and preserves orientation in the interior of the complement of $\Delta$.

Let $\Gamma$ be a Fuchsian group, operating on $U_1$, where $f_1 : U_1 \to \Delta$ is the universal covering, and $\Gamma$ is the lifting of $G$; that is, $U_1/\Gamma \cong \Delta/G$. Let $\Gamma^*$ be the $Z_2$-extension of $\Gamma$ which covers $G \cup J$.

Suppose that no orientation-reversing $\gamma^* \in \Gamma^*$ had a fixed (non-Euclidean) line in $U_1$. Then for every such $\gamma^*$, $(\gamma^*)^2 = \gamma \in \Gamma$, and $A_\gamma$, the axis of $\gamma$ is invariant under $\gamma^*$. Choose $\gamma_0^*$ to minimize the non-Euclidean length of $A_{\gamma_0}/\Gamma$. Then since $\gamma^*$ projects onto an involution, $A_{\gamma_0}/\Gamma$ can have at most one double point. One double point would lift to a fixed point of some $\gamma^*$. Hence $A_{\gamma_0}/\Gamma$ is a simple loop. Since $J^2 = 1$, $A_{\gamma_0}$ projects onto a simple loop in $\Delta$. This simple loop is invariant under $J$; hence $J$ interchanges the two topological discs bounded by the loop. Finally, since $J$ is the identity on $\Lambda(G)$, $\Lambda(G) = \emptyset$ — contradicting the assumption that $G$ is nonelementary. We conclude that some lifting $\gamma^*$ of $J$ has a line of fixed points in $U_1$; hence $J$ has fixed points in $\Delta$.

The set $T$ of fixed points of $J$ must divide $\Delta$ into at least two regions, for if not, we could repeat the above argument looking at the universal covering of $\Delta - T$. If there were more than two regions, we could as above define $J^* = J$ in two of these regions, and $J^* = 1$ elsewhere, to get a contradiction. Let $\Delta_1$ and $\Delta_2$ be the components of $\Delta - T$. Since
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\[ gT = T \text{ for all } g \in G, H, \text{ the subgroup of } G \text{ keeping } \Delta_1 \text{ invariant is of index at most } 2 \text{ in } G. \] 

The group \( H \) has invariant open sets \( \Delta_1 \) and \( \Delta_2 \), hence \( \Delta_1 \) and \( \Delta_2 \) are both simply connected.

Let \( f_1 : U_1 \to \Delta_1 \) be the Riemann map, and let \( \Gamma = f_1 H f_1^{-1} \) be the Fuchsian equivalent of \( H \). Define \( f_2 : U_2 \to \Delta_2 \) by \( f_2 = J \circ f_1 \circ j \). By Theorem B, \( f_1 \) and \( f_2 \) are restrictions of a fractional linear transformation \( f \). Then \( J = f \circ j \circ f^{-1} \).

**Proof of Theorem C.** If \( f_1(U_1) \cap f_2(U_2) = \emptyset \), then define

\[
\begin{align*}
J(z) &= f_2 \circ j \circ f_1^{-1}(z), & z & \in f_1(U_1), \\
&= f_1 \circ j \circ f_2^{-1}(z), & z & \in f_2(U_2), \\
&= z, & z & \in \Lambda(G).
\end{align*}
\]

Note \( f_1(U_1) \) and \( f_2(U_2) \) are both invariant under \( G \), and so, upon addition of some isolated points, are both simply-connected.

Since \( f_i(U_i) \), \( i = 1, 2 \), is, except for countably many isolated points, a component of \( G \), if \( f_1(U_1) \cap f_2(U_2) \neq \emptyset \), then (modulo some isolated points) \( f_1(U_1) = f_2(U_2) \). In this case, set

\[
\begin{align*}
J(z) &= f_2 \circ j \circ f_1^{-1}(z), & z & \in f_1(U_1), \\
&= z, & z & \in \Lambda(G).
\end{align*}
\]

It is obvious that each of the maps \( J \) defined above extend by continuity to the isolated points at which they have not yet been defined.

**Proof of Theorem D.** By Accola's remark [1], \( \Delta_1 \) and \( \Delta_2 \) are both simply-connected. Let \( F_1 : \Delta_1 \to U_1 \) be the Riemann map, and let \( \psi : G \to \Gamma \) be the isomorphism of \( G \) onto the Fuchsian group \( \Gamma \) given by \( \psi(g) = F_1 \circ g \circ F_1^{-1} \). Using the Fenchel-Nielsen Isomorphism Theorem [5] (see, for example, Marden [8] for a proof) there is a homeomorphism \( F_2 : \Delta_2 \to U_2 \) with \( F_2 \circ g \circ F_2^{-1} = \psi(g) \) for all \( g \in G \). By Ahlfors' Finiteness Theorem [2], and Bers' Approximation Theorem [4], \( F_2 \) can be chosen to be quasiconformal. Set

\[
\mu(z) = \frac{\partial F_2/\partial z}{\partial F_2/\partial \bar{z}}, \quad z \in \Delta_2,
\]

\[
= 0, \quad z \notin \Delta_2,
\]

and let \( w^\# \) (see Ahlfors-Bers [3]) be a quasiconformal homeomorphism satisfying

\[
\frac{\partial w^\#/\partial z}{\partial w^\#/\partial \bar{z}} = \mu \frac{\partial w^\#/\partial z}{\partial w^\#/\partial \bar{z}}.
\]
Then $G^\mu = w^\mu G(w^\mu)^{-1}$ is again a Kleinian group and $w^\mu \circ (F_i)^{-1}$ is conformal in $U_i$, $i = 1, 2$. Hence $J = w^\mu \circ {F_i}^{-1} \circ j \circ F_i \circ (w^\mu)^{-1}$ is an anti-conformal homeomorphism of $\Omega(G^\mu)$ which commutes with every element of $G^\mu$. By Theorem A, the group $G^\mu$, which is a quasiconformal deformation of $G$, is Fuchsian.

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