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PURE, NORMAL MAXIMAL SUBFIELDS FOR DIVISION ALGEBRAS IN THE SCHUR SUBGROUP

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In this paper we shall give very simple relations which define those division algebras contained in the Schur subgroup of the Brauer group of a field F of characteristic zero.

Any irreducible representation of a finite group over a field of characteristic zero corresponds to a simple component $\mathfrak{A}$ in the group algebra over that field. The character afforded by the representation will have as constituent an irreducible complex character χ, and the center F of $\mathfrak{A}$ will contain the values of χ on the group. Decompose $\mathfrak{A}$ as $\mathfrak{D}\otimes\mathfrak{M}$, where $\mathfrak{D}$ is a division algebra and $\mathfrak{M}$ is a full matrix algebra, both with center F. All of the finite-dimensional division algebras with center F form an Abelian group, called the Brauer group of F. Those division algebras obtained by the method just described form a subgroup, the Schur subgroup of F. The dimension of $\mathfrak{D}$ over F is the square of an integer m which is called the Schur index of χ. It has recently been proved by M. Benard and M. Schacher [2] that F contains the mth roots of unity. The purpose of this note is to draw attention to the following interesting consequence of this result.

THEOREM. *Let $\mathfrak{D}$ be the division algebra appearing in the factorization of a simple component of the group algebra of a finite group over a field of characteristic zero. Then $\mathfrak{D}$ is generated over its center F by elements A and B satisfying the relations*

(1) $A^{-1}B^{-1}AB = \varepsilon$,

(2) $A^m \in F$,

(3) $B^m \in F$,

where m is the index of $\mathfrak{D}$ and ε is a primitive mth root of unity.

The fields $K = F(A)$ and $L = F(B)$ are pure maximal subfields of $\mathfrak{D}$ which are normal extensions of F with cyclic Galois group.

We begin our proof with the observation that there is a division algebra $\mathfrak{D}_0$ in the rational group algebra such that $\mathfrak{D} = \mathfrak{D}_0 \otimes F$. This is a tensor product over the center $Q(\chi)$ of $\mathfrak{D}_0$ where χ is some irreducible complex character afforded by the group. The fundamental structure theorem [1,

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Theorem 32, p. 149] asserts that $\mathfrak{D}_0$, a rational division algebra, contains a maximal subfield which is a normal extension of $Q(\chi)$ and has a cyclic Galois group over $Q(\chi)$. Therefore $\mathfrak{D}$ contains a maximal subfield K which is a normal extension of F of dimension m, with a cyclic Galois group over F. From field theory, we know K is a simple extension of F, obtained by the adjunction of a single element α to F.

Since F contains a primitive mth root of unity ε, a fundamental result in Galois theory asserts that K is a pure extension of F. To recall a proof of this result, suppose σ generates the Galois group of K over F. For each power α^i of α, form the Lagrange resolvent

$$A_i = \alpha^i + (\alpha^i)^\sigma\varepsilon^{-1} + (\alpha^i)^{\sigma^2}\varepsilon^{-2} + \ldots + (\alpha^i)^{\sigma^{m-1}}\varepsilon^{-(m-1)}.$$

It can be shown that for some integer i, A_i is not zero. Call this $A_i = A$. From the definition of A we have

$$A^\sigma = A\varepsilon.$$

From this it follows that A has m distinct conjugates under the powers of σ, and therefore A generates K over F. The norm of A with respect to σ is

$$\begin{aligned} AA^\sigma A^{\sigma^2}\cdots A^{\sigma^{m-1}} &= AA\varepsilon A\varepsilon^2\cdots A\varepsilon^{m-1} \\ &= A^m\varepsilon^{1+2+\cdots+m-1} \\ &= A^m\varepsilon^{m(m-1)/2} = \pm A^m. \end{aligned}$$

Since this product is fixed under σ, we conclude $A^m \in F$. Thus $K = F(A)$ is a pure, cyclic normal extension of F.

We next show the existence of an element B in $\mathfrak{D}$ for which $B^{-1}AB = A^\sigma$ or

$$A^{-1}B^{-1}AB = \varepsilon. \tag{1}$$

This follows from two theorems about simple algebras. The first [**1**, Theorem 14, Lemma 2, p. 55] implies that the automorphism σ of K can be extended to an automorphism of $\mathfrak{D}$. The Skolem-Noether theorem [**1**, Theorem 5, p. 51] which asserts that any automorphism of $\mathfrak{D}$ is inner proves the existence of B.

Since B^m centralizes K, a maximal subfield, K must contain B^m. Since B^m is centralized by B, it is invariant under σ and must lie in F. We may invert the relationship (1) to get

$$A^{-1}BA = B\varepsilon^{-1}.$$

Therefore conjugation by A is an automorphism of the field $L = F(B)$ of order m fixing F. Hence the dimension $[L:F]$ is at least m. However m

is the maximum possible dimension of a subfield of $\mathfrak{D}$ over F. Thus L is a maximal subfield of $\mathfrak{D}$ of dimension m over F, hence a normal extension of F. We have now shown that K and L are pure maximal subfields of $\mathfrak{D}$ which are normal, cyclic extensions of F.

The proof that the mth roots of unity lie in F given in [2] makes use of a partial result proved by the author in [3]. An elementary and very elegant short proof of this result has just been given by G. J. Janusz [4].

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