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PUNCTUAL HILBERT SCHEMES[1]

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This note is about the structure of families of open ideals in the ring of power series in two variables. The Hilbert scheme parametrizing them is stratified into locally closed subschemes Z_T, whose dimension we calculate. We then discuss some global consequences for families of 0-dimensional schemes on a surface (§1, Corollaries 2, 3). Except in low characteristics, Z_T is locally an affine space (Theorem 2) and is a locally trivial bundle over the complete variety G_T parametrizing graded ideals of type T (Theorem 3).

1. **A stratification of the Hilbert scheme.** Let R be the ring of power series $k[[x_1, \ldots, x_r]]$ in r variables over an algebraically closed field k, with maximal ideal m; and let R_j denote the space of forms of degree j in R, so that $R = \prod R_j$, $j = 0, \ldots, \infty$. If I is an ideal in R, we let I_j denote the space of forms in R_j which are initial forms of elements of I. By the *type of* I we mean the sequence

$$(1) \qquad T(I) = (t_0, t_1, \ldots, t_j, \ldots), \quad \text{where } t_j = \dim_k(R_j/I_j).$$

We will sometimes refer to a type T, meaning a specific infinite sequence $(t_0, t_1, \ldots)$. By the *length* $|T|$ of T we mean $\sum t_j$, if it is finite. The *initial degree of* I is the smallest j for which $I_j \neq 0$. It depends only on the type of I. It is easy to show that if I has finite colength n, then $n = |T(I)|$, and $t_j = 0$ if $j \geqq n$.

Let $\mathrm{Hilb}^n R$ be the Hilbert scheme parametrizing the family of ideals of colength n in R, and Z_T the subscheme parametrizing ideals of a given type T where $|T| = n$. Then we get a stratification (see [7])

$$\mathrm{Hilb}^n R = \bigcup_{|T| = n} Z_T.$$

For the rest of the paper we consider the case $r = 2$, and let $A = k[[x, y]]$. If $I \subset A$ has colength n and initial degree d, then

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$$(2)\qquad \begin{gathered} T(I) = (1, 2, \ldots, d, t_d, \ldots, t_{n-1}, 0, 0, \ldots), \\ 0 \leqq t_j \leqq j + 1, \sum t_j = n \text{ and } d \geqq t_d \geqq \cdots \geqq t_{n-1}. \end{gathered}$$

The above conditions (2) characterize the sequences that occur as types of an open ideal in A. We assume T is a sequence satisfying (2), and set

$$(3)\qquad \begin{aligned} e_j &= t_{j-1} - t_j \quad \text{if } j \geqq d, \\ &= 0 \qquad\qquad \text{otherwise}. \end{aligned}$$

We call e_j the *jump index of* T. We let G_T be the subscheme of $\mathrm{Hilb}^n R$ parametrizing the graded ideals of type T.

THEOREM 1.

$$(4)\qquad \begin{gathered} \dim Z_T = n - \sum e_j(e_j + 1)/2 = n - d - \sum e_j(e_j - 1)/2, \\ \dim G_T = \sum_{j \geqq d} (e_j + 1)e_{j+1}. \end{gathered}$$

The proof is by an induction on the colength n. We compare ideals I of type T having a certain weak normal form with the ideals $I:x$, which have a type $T:x$ which depends only on T and is of length $< n$.

Let $T_n = (1, 1, \ldots, 1, 0, 0, \ldots)$, where $|T_n| = n$ and $n > 1$.

COROLLARY 1.

$$\begin{aligned} \dim Z_T = n - 1 &= \dim \mathrm{Hilb}^n A \quad && \text{if } T = T_n, \\ < n - 1 & && \text{if } T \neq T_n, |T| = n. \end{aligned}$$

PROOF. If $d = 1$, then $T = T_n$ by (2) and $\dim Z_T = n - 1$ by (4). If $d > 1$ then $\dim Z_T \leqq n - d \leqq n - 2$.

Suppose X is a nonsingular surface, and consider the Chow morphism (see [**2**]) onto the n-fold symmetric product $X^{(n)}$:

$$w_n : \mathrm{Hilb}^n X \to X^{(n)}.$$

$\mathrm{Hilb}^n X$ is a desingularization of $X^{(n)}$. If z is a geometric point of $X^{(n)}$ representing the zero-cycle $W = \Sigma n_i Q_i$, of degree n on X, then $w_n^{-1}(z) \simeq \prod \mathrm{Hilb}^{n_i} A$ and $w_n^{-1}(z)$ parametrizes the subschemes of X having cycle W (see [**2**]). This shows

COROLLARY 2. $\dim w_n^{-1}(z) = \sum_{i=1}^{r} (n_i - 1) = n - r$.

Let π denote the partition $(n_1, \ldots, n_r)$ of n, and let X_π be the subvariety of $X^{(n)}$ parametrizing cycles of index π, and $Y_\pi = w_n^{-1}(X_\pi)$. Then by Corollary 2,

(5) $$\dim Y_\pi = n + r \quad \text{and} \quad \operatorname{cod} Y_\pi = n - r.$$

Let D denote the singular locus on $X^{(n)}$, parametrizing cycles $\Sigma n_i Q_i$ where some $n_i \neq 1$, and let $B = w_n^{-1}(D)$. B is the branch locus of the universal subscheme Z^n,

$$X \times \mathrm{Hilb}^n X \supset Z^n \to \mathrm{Hilb}^n X,$$

over $\mathrm{Hilb}^n X$ (see [**2**] or [**1**]).

COROLLARY 3. *B is irreducible.*

PROOF. By (5), the highest dimensional component of B is just $Y_{[2]}$ mod a lower dimensional subvariety, where [2] denotes the partition $(2, 1, 1, \ldots)$ of n. But $X_{[2]}$ is irreducible, and the fibers of $w_n: Y_{[2]} \to X_{[2]}$ are projective lines, so $Y_{[2]}$ is irreducible. Z^n is flat over the nonsingular $\mathrm{Hilb}^n X$, so it is Cohen-Macauley. It must be nonsingular in codimension 1, since any singularity would have to be over $Y_{[2]}$ and we can reduce to the case $\mathrm{Hilb}^2 X$ where Z^2 is nonsingular. Therefore Z^n is normal, and by purity of branch locus, B is pure codimension 1. Corollary 3 follows from the irreducibility of $Y_{[2]}$.

From Corollary 3 one deduces that $\mathrm{Pic}(\mathrm{Hilb}^n X) \otimes Q = Q \oplus \mathrm{Pic}\, X^{(n)} \otimes Q$.

2. **The varieties Z_T and G_T**, $r = 2$. We assume $|T| = n$. Except where noted, the results are valid in all characteristics.

THEOREM 2. *Z_T and G_T each have a connected cover by Zariski opens in an affine space, hence they are irreducible, rational, and nonsingular. G_T is also complete.*

THEOREM 2′. *If* $\operatorname{char} k = 0$, *or* $\operatorname{char} k > n$, *$Z_T$ and G_T each have a connected cover by open sets isomorphic to affine spaces.*

The proof uses a normal form for ideals of type T in A. Theorem 2′ arises from a better normal form in those cases.

To each ideal I we associate the completed graded ideal $\mathrm{gr}(I) = \prod I_j$, and clearly $T(I) = T(\mathrm{gr}\, I)$. This leads to a morphism $\pi: Z_T \to G_T$, having a section $s: G_T \to Z_T$ induced by the inclusion of graded ideals of type T in all ideals of type T.

THEOREM 3. *$\pi: Z_T \to G_T$ is a locally trivial bundle having fibre an affine space and having a natural "0-section" s. In general, Z_T is not an algebraic vector bundle over G_T.*

These are bundles with group $\mathrm{Aut}(A)$, where A = affine space, but whose group cannot in general be reduced to $Gl(A)$. In characteristic 0, such a bundle is diffeomorphic (but not algebraically isomorphic) to a vector bundle. See the example below.

The structure of G_T is known only in the simplest cases; for example, if T satisfies $e_j(e_{j+1}) = 0$ for all j, then $G_T = \prod_{j>d} \boldsymbol{P}_{e_j}$.

EXAMPLE. If $T = T_n$, then Z_T parametrizes ideals in A of colength n, and initial degree 1. Typical such ideals are

$$I_C = (y + c_0 x + \dots + c_{n-2} x^{n-1}, m^n)$$

and

$$I_B = (x + b_0 y + \dots + b_{n-2} y^{n-1}, m^n).$$

Thus $\mathrm{gr}(I_C) = (y + c_0 x, m^n)$, and $\mathrm{gr}(I_B) = (x + b_0 y, m^n)$.

Every ideal of type T_n has one or both of the above forms; c_0 and b_0 are coordinates on the two affine pieces of $P_1 = G_T$; and the analysis of Z_n proceeds from an analysis of the transition functions:

$$I_B = I_C \Leftrightarrow \left\{ \begin{array}{c} b_0 = g_0(c_0^{-1}) = c_0^{-1} \\ \cdots \\ b_j = g_j(c_0^{-1}, c_1, \dots, c_j) \\ \cdots \end{array} \right\}.$$

There is a natural projection from ideals of type T_n to ideals of type $T_{n-1}: I \to I + m^{n-1}$ or "ignore the last coefficient," hence a projection $p_n: Z_n \to Z_{n-1}$, and there is an exact sequence of bundles over P_1,

$$0 \to \mathcal{O}(n) \to Z_n \xrightarrow{p_n} Z_{n-1} \to 0, \tag{6}$$

exact in the sense that $\mathcal{O}(n) = p_n^{-1}(s(P_1))$.

By (6), $Z_3 = \mathcal{O}(3)$ over P_1. However in **[8]** we show that if $n > 3$, then Z_n is not a vector bundle.

The group of substitutions with determinant one, $(\mathrm{Aut}(A/m^n))$, acts on ideals in A/m^n and preserves the type, so $\mathrm{Aut}(A/m^n)$ acts on Z_T if $|T| \leqq n$. The action is clearly transitive on Z_n, and the isotropy group of the point $p \in Z_n$ corresponding to $I = (x, m^n)$ is the triangular subgroup Δ_n of substitutions $x \to u^{-1}x$, $y \to uy + ax$ with $u \in (A/m^n)^*$ and $a \in (A/m^{n-1})$. Thus $Z_n = \mathrm{Aut}(A/m^n)/\Delta_n$, a quotient variety for the left cosets of Δ_n. This example generalizes to $r > 2$, $G_T = P_r$.

In general $Sl_2(k) = Sl_2(A/m)$ acts on G_T, and the kernel K,

$$1 \to K \to \mathrm{Aut}(A/m^n) \to Sl_2(A/m) \to 1,$$

acts on the fibre of Z_T over G_T if $|T| = n$, and neither action is transitive. The analysis of this action will certainly lead to deeper results.

The dimension results of §1 and the irreducibility of Z_T given in Theorem 2 support the long standing

CONJECTURE 1. $\mathrm{Hilb}^n A$ is irreducible, $\mathrm{Hilb}^n A = \bar{Z}_n$, the completion of Z_n.

This conjecture is implied by

CONJECTURE 2. Z_T is in the singular locus of $\mathrm{Hilb}^n A$ if $T \neq T_n$.

We have checked Conjecture 2 if T has initial degree 2 and in some other cases—in fact all cases where we have specific parameters for ideals near a given ideal I of colength n in A. The analogous conjectures are false in higher dimensions [9].

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