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FOURIER COEFFICIENTS OF CERTAIN EISENSTEIN SERIES[1]

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Let K be a field of characteristic $\neq 2, 3$ and let $\mathfrak{J}_K$ be the exceptional Jordan algebra of dimension 27 consisting of hermitian 3×3 matrices with entries in the Cayley-Dickson algebra $\mathfrak{C}_K$. The product $X \circ Y$ in $\mathfrak{J}$ is $\frac{1}{2}(XY + YX)$, where XY is the matrix product. In [3], there are defined a norm (det) and a trace (tr) on $\mathfrak{J}$. Let $(\ ,\ ,\ )$ be the symmetric trilinear form on $\mathfrak{J} \times \mathfrak{J} \times \mathfrak{J}$ such that $(A, A, A) = \det(A)$, and define a bilinear map $\mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$, which takes (A, B) to $A \times B$, by requiring that $(A \times B, C) = 3(A, B, C)$ for each $C \in \mathfrak{J}$, where $(X, Y) = \operatorname{tr}(X \circ Y)$. Then $A \times A$ plays the role of the matrix adjoint of A, and the notions just introduced can be used to define the rank of each element $A \in \mathfrak{J}$. We denote this by $\operatorname{rk}(A)$. In particular, $\operatorname{rk}(A) = 3$ if and only if $\det(A) \neq 0$. Let $\mathfrak{k}_j = \{A \in \mathfrak{J}_R : \operatorname{rk}(A) = j\}$. The tube domain associated to $\mathfrak{J}$ is

$$\mathfrak{T} = \{Z = X + iY \in \mathfrak{J}_C : Y \in \mathfrak{k}_3^+\},$$

where $\mathfrak{k}_j^+ = \{Y \in \mathfrak{k}_j : Y = X^2 \text{ for some } X \in \mathfrak{J}_R\}$.

The group of holomorphic automorphisms of $\mathfrak{T}$ is isogenous to a certain algebraic Q-group which is of type E_7. Baily [1] has defined an arithmetic subgroup Γ of G_Q which is a unicuspidal subgroup of G and a maximal discrete subgroup of G_R. Let $J(Z, \gamma)$ be the functional determinant of γ at $Z, Z \in \mathfrak{T}$. Let Γ_0 be the subgroup of Γ which stabilizes a certain zero-dimensional rational boundary component $\mathfrak{T}_0^\infty$ of $\mathfrak{T}$, as in [1, §7]. We let

$$E_g(Z) = \sum_{\gamma \in \Gamma/\Gamma_0} J(Z, \gamma)^{g/18},$$

where $g \equiv 0 \pmod{36}$ and $g > 19$. Then the Eisenstein series E_g is an automorphic form of weight $g/18$ with respect to the group Γ and the factor of automorphy J. It has an absolutely convergent Fourier expansion

$$E_g(Z) = \sum_{T \in \Lambda^+} a_g(T) e^{2\pi i (T, Z)},$$

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[1] This paper describes a portion of the author's doctoral thesis written under the direction of Professor Walter L. Baily, Jr. at the University of Chicago.

where Λ^+ is the intersection of a certain lattice in $\mathfrak{J}_R$ with the set of squares in $\mathfrak{J}_R$. The main result of [1] is that $a_g(T) \in Q$ for each $T \in \Lambda^+$.

For any $T \in \mathfrak{J}_Q$ one can define three numerical invariants, the "elementary divisors of T." We call their respective p-adic orders the "p-adic order invariants of T." Let $\det_j(T)$ be the product of the first j elementary divisors. Then $\det_3(T) = \det(T)$ and if $\mathrm{rk}(T) = j$, then $\det_j(T) \neq 0$. Let Υ_j be the 3×3 matrix havings 1's in the topmost j positions on the diagonal and zeros elsewhere. The nth Bernoulli number B_n is defined by the symbolic recursion process $B_n \to B^n$, $(1 + B)^{n+1} - B^{n+1} = 0$, $B_0 = 1$. In particular, $B_{2n+1} = 0$ if $n \geqq 1$. The purpose of this note is to announce the following result.

THEOREM. *For any $T \in \Lambda^+ \cap \mathfrak{k}_j$ with $j = 0, 1, 2, 3$,*

$$a_g(T) = a_g(\Upsilon_j)\det_j(T)^{g+3-4j} \prod_{p | \det_j(T)} f_T^p(p^{4j-3-g}),$$

where

$$a_g(\Upsilon_j) = 2^{j(2j-1)} \prod_{n=0}^{j-1} \left\{ \frac{g - 4n}{B_{g-4n}} \right\},$$

and where f_T^p is a monic polynomial with rational integer coefficients and with degree $D = \mathrm{ord}_p(\det_j(T))$. Furthermore, f_T^p is determined by the p-adic order invariants of T; hence, for fixed g, $a_g(T)$ depends only on the elementary divisors of $T \in \Lambda^+$.

Let $\| \ \|_p$ be the ordinary p-adic absolute value. Then $\|\det_j(T)\|_p^{4j-3-g} f_T^p(p^{4j-3-g})$ is a rational integer. The Fourier coefficients $a_g(T)$, for fixed g, are integral multiples of $a_g(\Upsilon_j)$, where $j = \mathrm{rk}(T)$. Note that $a_g(\Upsilon_j) \in Q$.

COROLLARY. *Let δ_g be the product of the numerators of the rational numbers B_{g-4n}, where $n = 0, 1, 2$. Then the Γ-automorphic form $\delta_g E_g$ has rational integer Fourier coefficients.*

Suppose that $T \in \Lambda^+ \cap \mathfrak{k}_2$ and that the order invariants of T are τ, τ' where $\tau \leqq \tau'$. Then $f_T^p(X) = \sum_{k=0}^{\tau} p^{4k} \sum_{m=k}^{\tau+\tau'-k} X^m$. We have not determined f_T^p so explicitly when $\mathrm{rk}(T) = 3$, but it is easy to compute individual examples from our work. For example, when $T = p\Upsilon_3$, we have

$$f_T^p(X) = X^3 + (p^8 + p^4 + 1)X^2 + (p^8 + p^4 + 1)X + 1$$

Similar but essentially less precise results have been obtained in the case of the group $Sp_n(Z)$ acting on the Siegel upper half-space $\mathfrak{H}_n$ of rank n by Maass [4] when $n = 2$, by Siegel [5], and by Eichler [2]. Both Maass and Eichler used the theory of Hecke operators, while Siegel relied on the analytic theory of quadratic forms. By contrast, our methods are entirely elementary.

References

1. W. L. Baily, Jr., *An exceptional arithmetic group and its Eisenstein series,* Ann. of Math. (2) **91** (1970), 512–549. MR **42** #4674.

2. M. Eichler, *Zur Begründung der Theorie der automorphen Funktionen in mehreren Variablen,* Aequationes Math. **3** (1969), 93–111. MR **41** #2056.

3. H. Freudenthal, *Zur ebenen Oktavengeometrie,* Nederl. Akad. Wetensch. Proc. Ser. A **56** = Indag. Math. **15** (1953), 195–200. MR **15**, 56.

4. H. Maass, *Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades,* Mat.-Fys. Medd. Danske Vid. Selsk. **34** (1964), no. 7, 25 pp. MR **30** #1985.

5. C. L. Siegel, *Über die Fourierschen Koeffizienten der Eisensteinschen Reihen,* Mat.-Fys. Medd. Danske Vid. Selsk. **34** (1964), no. 6, 20 pp. MR **30** #1984.

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