FOURIER COEFFICIENTS OF CERTAIN EISENSTEIN SERIES

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Communicated by Alberto Calderón, February 28, 1972

Let $K$ be a field of characteristic $\neq 2, 3$ and let $\mathfrak{S}_K$ be the exceptional Jordan algebra of dimension 27 consisting of hermitian $3 \times 3$ matrices with entries in the Cayley-Dickson algebra $\mathfrak{C}_K$. The product $X \circ Y$ in $\mathfrak{S}$ is $\frac{1}{2}(XY + YX)$, where $XY$ is the matrix product. In [3], there are defined a norm (det) and a trace (tr) on $\mathfrak{S}$. Let $(\ , \ , )$ be the symmetric trilinear form on $\mathfrak{S} \times \mathfrak{S} \times \mathfrak{S}$ such that $(A, A, A) = \det(A)$, and define a bilinear map $\mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$, which takes $(A, B)$ to $A \times B$, by requiring that $(A \times B, C) = 3(A, B, C)$ for each $C \in \mathfrak{S}$, where $(X, Y) = \text{tr}(X \circ Y)$. Then $A \times A$ plays the role of the matrix adjoint of $A$, and the notions just introduced can be used to define the rank of each element $A \in \mathfrak{S}$. We denote this by $\text{rk}(A)$. In particular, $\text{rk}(A) = 3$ if and only if $\det(A) \neq 0$. Let $\mathfrak{I}_j = \{A \in \mathfrak{S}_K : \text{rk}(A) = j\}$. The tube domain associated to $\mathfrak{S}$ is

\[ \mathfrak{T} = \{Z = X + iY \in \mathfrak{S}_C : Y \in \mathfrak{I}_3^+\}, \]

where $\mathfrak{I}_3^+ = \{Y \in \mathfrak{I}_3 : \text{tr}(X^2) = 0 \text{ for some } X \in \mathfrak{S}_K\}$.

The group of holomorphic automorphisms of $\mathfrak{T}$ is isogenous to a certain algebraic $\mathbb{Q}$-group which is of type $E_7$. Baily [1] has defined an arithmetic subgroup $\Gamma$ of $G_\mathbb{Q}$ which is a unicuspidal subgroup of $G_\mathbb{R}$ and a maximal discrete subgroup of $G_\mathbb{R}$. Let $J(Z, \gamma)$ be the functional determinant of $\gamma$ at $Z, Z \in \mathfrak{T}$. Let $\Gamma_0$ be the subgroup of $\Gamma$ which stabilizes a certain zero-dimensional rational boundary component $\mathfrak{T}^\infty_0$ of $\mathfrak{T}$, as in [1, §7]. We let

\[ E_g(Z) = \sum_{\gamma \in \Gamma \Gamma_0} J(Z, \gamma)^{g/18}, \]

where $g \equiv 0(\text{mod } 36)$ and $g > 19$. Then the Eisenstein series $E_g$ is an automorphic form of weight $g/18$ with respect to the group $\Gamma$ and the factor of automorphy $J$. It has an absolutely convergent Fourier expansion

\[ E_g(Z) = \sum_{T \in \Lambda^*} a_g(T)e^{2\pi i(T, Z)}, \]

AMS 1970 subject classifications. Primary 10D20; Secondary 20G30.

Key words and phrases. Fourier coefficients, Eisenstein series, algebraic $\mathbb{Q}$-group of Type $E_7$, exceptional Jordan algebra, arithmetic subgroup.

This paper describes a portion of the author's doctoral thesis written under the direction of Professor Walter L. Baily, Jr. at the University of Chicago.
where $\Lambda^+$ is the intersection of a certain lattice in $\mathbb{H}_n$ with the set of squares in $\mathbb{H}_n$. The main result of [1] is that $a_g(T) \in \mathcal{Q}$ for each $T \in \Lambda^+$.

For any $T \in \mathbb{H}_n$, one can define three numerical invariants, the “elementary divisors of $T$.” We call their respective $p$-adic orders the “$p$-adic order invariants of $T$.” Let $\det_j(T)$ be the product of the first $j$ elementary divisors. Then $\det_3(T) = \det(T)$ and if $\text{rk}(T) = j$, then $\det_j(T) \neq 0$. Let $Y_j$ be the $3 \times 3$ matrix having $1$’s in the topmost $j$ positions on the diagonal and zeros elsewhere. The $n$th Bernoulli number $B_n$ is defined by the symbolic recursion process

$$B_n \rightarrow B_{n+1}^{(1)} + B_{n-1}^{(1)} = 0, \quad B_0 = 1.$$ 

In particular, $B_{2n+1} = 0$ if $n \geq 1$. The purpose of this note is to announce the following result.

**Theorem.** For any $T \in \Lambda^+ \cap \mathfrak{I}_j$ with $j = 0, 1, 2, 3$,

$$a_g(T) = a_g(Y_j) \det_j(T)^{s+3-4j} \prod_{p \mid \det_j(T)} f_p^j(p^{4j-3-s}),$$

where

$$a_g(Y_j) = 2^{j(2j-1)} \sum_{n=0}^{j-1} \left\{ \frac{g - 4n}{B_g - 4n} \right\},$$

and where $f_p^j$ is a monic polynomial with rational integer coefficients and with degree $D = \text{ord}_p(\det_j(T))$. Furthermore, $f_p^j$ is determined by the $p$-adic order invariants of $T$; hence, for fixed $g$, $a_g(T)$ depends only on the elementary divisors of $T \in \Lambda^+$.

Let $\| \|_p$ be the ordinary $p$-adic absolute value. Then $\|\det_j(T)\|_p^{4j-3-s} f_p^j(p^{4j-3-s})$ is a rational integer. The Fourier coefficients $a_g(T)$, for fixed $g$, are integral multiples of $a_g(Y_j)$, where $j = \text{rk}(T)$. Note that $a_g(Y_j) \in \mathcal{Q}$.

**Corollary.** Let $\delta_g$ be the product of the numerators of the rational numbers $B_{g-4n}$, where $n = 0, 1, 2$. Then the $\Gamma$-automorphic form $\delta_g E_g$ has rational integer Fourier coefficients.

Suppose that $T \in \Lambda^+ \cap \mathfrak{I}_2$ and that the order invariants of $T$ are $\tau, \tau'$ where $\tau \leq \tau'$. Then $f_p^j(X) = \sum_{k=0}^{j} p^{4k} \sum_{m}^{\tau'+1-k} X^m$. We have not determined $f_p^j$ so explicitly when $\text{rk}(T) = 3$, but it is easy to compute individual examples from our work. For example, when $T = p\mathfrak{I}_3$, we have

$$f_p^j(X) = X^3 + (p^8 + p^4 + 1)X^2 + (p^8 + p^4 + 1)X + 1$$

Similar but essentially less precise results have been obtained in the case of the group $Sp_4(\mathbb{Z})$ acting on the Siegel upper half-space $\mathbb{H}_n$ of rank $n$ by Maass [4] when $n = 2$, by Siegel [5], and by Eichler [2]. Both Maass
and Eichler used the theory of Hecke operators, while Siegel relied on the analytic theory of quadratic forms. By contrast, our methods are entirely elementary.

REFERENCES


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