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# FOLIATIONS AND THE PROPAGATION OF ZEROES OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** Let $P(x, D)$ be a linear partial differential operator of order $m$ with complex-valued coefficients defined and analytic in an open connected set $\Omega$ in $R^n$,

$$P(x, D) = \sum_{|\alpha| \leq m} a^{\alpha}(x)D^{\alpha}, \tag{1}$$

with the usual notation. The principal part $P_m(x, D)$ is the homogeneous part of $P(x, D)$ of order $m$. At a fixed point $x \in \Omega$, the (real) zeroes of $P_m(x, \xi)$ form a cone in $R^n$ which is called the (real) characteristic cone of $P(x, D)$ at $x$. We will denote by $\mathscr{A}$ the ring of real-valued analytic functions in $\Omega$.

In this paper we consider partial differential operators having the following property: There exist $r$ analytic vector fields in $\Omega$,

$$A_j = \sum_{i=1}^{n} a_j^i D_i, \qquad j = 1, \ldots, r, \tag{2}$$

with $a_j^i \in \mathscr{A}, i = 1, \ldots, n, j = 1, \ldots, r$, such that at each point of $\Omega$ the characteristic cone of $P(x, D)$ is orthogonal to every $A_j$. More precisely, we assume that, for every $x \in \Omega$,

$$P_m(x, \xi) = 0, \quad \xi \in R^n \Rightarrow \sum_{i=1}^{n} a_j^i(x)\xi_i = 0, \qquad j = 1, \ldots, r. \tag{3}$$

We will denote by $\mathscr{L}(A_1, \ldots, A_r)$ the Lie algebra generated by $A_1, \ldots, A_r$, i.e. the smallest set of analytic vector fields in $\Omega$ which is closed under the operations of taking brackets and linear combinations with coefficients in $\mathscr{A}$.

According to a theorem of Nagano [1], the Lie algebra $\mathscr{L}(A_1, \ldots, A_r)$ defines a unique partition of $\Omega$ into maximal integral manifolds of $\mathscr{L}(A_1, \ldots, A_r)$, that is, $\Omega$ is the disjoint union of maximal integral manifolds of $\mathscr{L}(A_1, \ldots, A_r)$. This partition is called a foliation and each maximal integral manifold is called a leaf of the foliation.

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In this paper we prove that the zeroes of solutions of the equation $P(x, D)u = 0$ propagate along the leaves of the foliation defined by $\mathscr{L}(A_1, \ldots, A_r)$. More precisely let $u$ be a distribution solution of $P(x, D)u = 0$ in $\Omega$ and suppose that $u$ vanishes in an open neighborhood of a point $x \in \Omega$. Then $u$ also vanishes in an open neighborhood of every point of the leaf through $x$ of the foliation defined by $\mathscr{L}(A_1, \ldots, A_r)$.

This result includes the well-known result on the propagation of zeroes of solutions of elliptic equations. It includes also the result of Bony [2] concerning degenerate elliptic second order equations and the result of Zachmanoglou [3] concerning first order equations with complex-valued coefficients.

2. **Some results on foliations.** The bracket of two vector fields $A$ and $B$ is the commutator $[A, B] = AB - BA$. If $A$ and $B$ have analytic coefficients in $\Omega$, then $[A, B]$ is also a vector field with analytic coefficients in $\Omega$. The (real) vector space of all real analytic vector fields in $\Omega$, equipped with the bracket operation, is a Lie algebra denoted by $\mathscr{L}(\Omega)$. It is also a module over the ring $\mathscr{A}$. $\mathscr{A}(A_1, \ldots, A_r)$ will denote the smallest $\mathscr{A}$-submodule of $\mathscr{L}(\Omega)$ containing the vector fields $A_1, \ldots, A_r$. A vector subspace of $\mathscr{L}(\Omega)$ which is closed under the bracket operation is a Lie subalgebra of $\mathscr{L}(\Omega)$. $\mathscr{L}(A_1, \ldots, A_r)$ is the smallest $\mathscr{A}$-submodule and Lie subalgebra of $\mathscr{L}(\Omega)$ containing $A_1, \ldots, A_r$.

Let $\mathscr{L}$ be a vector subspace of $\mathscr{L}(\Omega)$. For any $x \in \Omega$ we set $\mathscr{L}(x) = \{A(x) : A \in \mathscr{L}\}$. $\mathscr{L}(x)$ is a subspace of $R^n$ and is called the integral element of $\mathscr{L}$ at $x$. An integral manifold $N$ of $\mathscr{L}$ is a connected submanifold of $\Omega$ such that, for every $x \in N$, the tangent space to $N$ at $x$ is equal to $\mathscr{L}(x)$.

THEOREM 1 (NAGANO). *If $\mathscr{L}$ is a Lie subalgebra of $\mathscr{L}(\Omega)$, then through every point $x \in \Omega$ passes a maximal integral manifold $L^x$ of $\mathscr{L}$. Any integral manifold of $\mathscr{L}$ containing $x$ is an open submanifold of $L^x$.*

According to Theorem 1, $\mathscr{L}$ defines a unique partition of $\Omega$ by maximal integral manifolds of $\mathscr{L}$ (that is, $\Omega$ is the disjoint union of maximal integral manifolds of $\mathscr{L}$). This partition of $\Omega$ will be called the foliation defined by $\mathscr{L}$ and each maximal integral manifold will be called a leaf of the foliation. Note that, for every $x \in \Omega$, the dimension of the leaf $L^x$ containing $x$ is equal to the dimension of the integral element $\mathscr{L}(x)$.

With the additional assumption that $\dim \mathscr{L}(x)$ is constant in $\Omega$, Theorem 1 is the classical theorem of Frobenius (see Chevalley [4]). However, the Frobenius theorem is also valid in the $C^\infty$ case. Theorem 1 was proved by Nagano [1] and it is not generally valid in the $C^\infty$ case.

An integral curve of an analytic vector field, say $A = \sum a^i D_i$, is a solution $x = x(t)$ of the system

$$dx_i/dt = a^i(x), \qquad i = 1, \ldots, n.$$

A trajectory of a collection $\mathscr{C}$ of analytic vector fields in $\Omega$ is a continuous, piecewise analytic curve in $\Omega$, each analytic piece of which is an integral curve of a member of $\mathscr{C}$.

THEOREM 2 (CHOW-HERMANN). *Let $\mathscr{L}$ be a (real) vector subspace of $\mathscr{L}(\Omega)$ and suppose that the smallest $\mathscr{A}$-submodule and Lie subalgebra of $\mathscr{L}(\Omega)$ containing $\mathscr{L}$ is equal to $\mathscr{L}(\Omega)$. Then, for every point $x \in \Omega$, the set of points of $\Omega$ that can be connected to $x$ by trajectories of $\mathscr{L}$ is equal to $\Omega$.*

A weaker form of this theorem was first proved by Chow [**5**]. A proof of Theorem 2 (with $\Omega$ being any differentiable manifold) is given in the book of Hermann [**6**, Chapter 18]. Incidentally Theorem 2 is also valid in the $C^\infty$ case.

We apply now Theorem 1 to the Lie subalgebra $\mathscr{L}(A_1, \ldots, A_r)$. The leaf of its foliation passing through the point $x$ will be denoted by $L^x(A_1, \ldots, A_r)$. We also apply Theorem 2 to the vector subspace $\mathscr{A}(A_1, \ldots, A_r)$ of $\mathscr{L}(\Omega)$. By restricting to the leaf $L^x(A_1, \ldots, A_r)$ Theorem 2 yields

THEOREM 3. *Let $x$ be any point of $\Omega$. The set of points of $\Omega$ that can be connected to $x$ by trajectories of $\mathscr{A}(A_1, \ldots, A_r)$, is equal to the leaf $L^x(A_1, \ldots, A_r)$.*

## 3. **The propagation of zeroes.**

LEMMA 1. *Let $A = \sum_{i=1}^n a^i D_i$ be an analytic vector field in $\Omega$ such that at each point of $\Omega$ the characteristic cone of $P(x, D)$ is orthogonal to $A$, that is, for every $x \in \Omega$,*

$$P_m(x, \xi) = 0, \quad \xi \in R^n \Rightarrow \sum_{i=1}^n a^i(x)\xi_i = 0.$$

*Let $u$ be a distribution solution of $P(x, D)u = 0$ in $\Omega$ and suppose that $u$ vanishes in an open neighborhood of some point $x$ of $\Omega$. Then $u$ vanishes in an open neighborhood of every point of the integral curve of $A$ passing through $x$.*

The proof of this lemma consists of locally straightening out the vector field $A$ and applying Theorem 1 of [**7**] concerning the propagation of zeroes of solutions of partial differential equations with flat characteristic cones. Theorem 1 of [**7**] was proved using Holmgren's uniqueness theorem as extended to distribution solutions by Hörmander [**8**] and a method first used by John [**9**].

It follows immediately from Lemma 1 that a solution of $P(x, D)u = 0$ which vanishes in a neighborhood of any point $x \in \Omega$ also vanishes in a neighborhood of every point of a trajectory of $\mathscr{A}(A_1, \ldots, A_r)$ containing $x$. Combining this with Theorem 3 we obtain the main result of this paper.

THEOREM 4. *Let $P(x, D)$ be a partial differential operator with analytic coefficients in an open connected set $\Omega$ of $R^n$. Suppose that there are $r$ analytic vector fields $A_1, \ldots, A_r$ in $\Omega$ such that at each point of $\Omega$ the characteristic cone of $P(x, D)$ is orthogonal to every $A_j$, i.e. for every $x \in \Omega$ condition* (3) *is satisfied. Let $u$ be a distribution solution of $P(x, D)u = 0$ in $\Omega$, and suppose that $u$ vanishes in an open neighborhood of some point $x \in \Omega$. Then $u$ vanishes in a neighborhood of every point of the leaf $L^x(A_1, \ldots, A_r)$ of the foliation defined by $\mathscr{L}(A_1, \ldots, A_r)$.*

COROLLARY 1. *Under the assumption of Theorem* 4 *and if, in addition, at every point of $\Omega$ the dimension of the integral element of $\mathscr{L}(A_1, \ldots, A_r)$ is equal to $n$, then every solution of $P(x, D)u = 0$ vanishing in a neighborhood of a point of $\Omega$ must vanish in the whole of $\Omega$.*

4. **Some examples.** If $P(x, D)$ is an elliptic operator in $\Omega$, then, by definition, for every $x \in \Omega$,

$$P_m(x, \xi) = 0, \quad \xi \in R^n \Leftrightarrow \xi = 0.$$

In this case we can take $A_j = D_j$, $j = 1, \ldots, n$. We have $\mathscr{L}(D_1, \ldots, D_n) = \mathscr{L}(\Omega)$ and its foliation consists of a single leaf, the whole of $\Omega$. Hence any solution of $P(x, D)u = 0$ vanishing in a neighborhood of a point of $\Omega$ must vanish in the whole of $\Omega$.

The operators studied by Bony [2] are of the form

$$P(x, D) = \sum_{j=1}^{r} A_j^2 + B + c$$

where $A_1, \ldots, A_r$ and $B$ are real first order operators with analytic coefficients in $\Omega$. At every point of $\Omega$ the characteristic cone is orthogonal to the vector fields $A_1, \ldots, A_r$. Under the assumption that the dimension of the integral element of $\mathscr{L}(A_1, \ldots, A_r)$ is equal to $n$ at every point of $\Omega$, Bony showed that any solution of $P(x, D)u = 0$ vanishing in a neighborhood of a point of $\Omega$ must vanish in the whole of $\Omega$. This of course is precisely the assertion of Corollary 1.

A first order operator with complex-valued coefficients is of the form

$$P(x, D) = A + iB + c$$

where $A$ and $B$ are real first order operators. The characteristic cone is orthogonal to the vector fields $A$ and $B$. The propagation of zeroes and uniqueness in the Cauchy problem for this operator were studied in [3].

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