1. Introduction. When Sinai [11], [12] and Bowen [1], [2] studied invariant measures for an Anosov diffeomorphism, or on basic sets for an Axiom A diffeomorphism, they encountered problems reminiscent of statistical mechanics (see [10, Chapter 7]). Sinai [13] has in fact explicitly used the techniques of statistical mechanics to show that an Anosov diffeomorphism does not in general have a smooth invariant measure.

We rewrite here a part of the general theory of statistical mechanics for the case of a compact set \( \Omega \) satisfying expansiveness and the specification property of Bowen [1]. Instead of a \( Z \) action we consider a \( Z^* \) action as is usual in lattice statistical mechanics (where \( \Omega = F^{Z^*} \) with \( F \) a finite set). This rewriting presents a number of technical problems, but the basic ideas are contained in the papers of Gallavotti, Lanford, Miracle-Sole, Robinson, and Ruelle [5], [7], [8], [9], etc.

2. Notation and assumptions. Given integers \( a_1, \ldots, a_v > 0 \), let \( Z^*(a) \) be the subgroup of \( Z^* \) with generators \((a_1, 0, \ldots, 0), \ldots, (0, \ldots, a_v)\). We write also

\[
\Lambda(a) = \{m \in Z^*: 0 \leq m_i < a_i\},
\]

\[
\Pi(a) = \{x \in \Omega: Z^*(a)x = \{x\}\}.
\]

If \((\Lambda_a)\) is a directed family of finite subsets of \( Z^* \), \( \Lambda_a \to \infty \) means \( \text{card} \Lambda_a \to \infty \) and \( \text{card}(\Lambda_a + F)/\text{card} \Lambda_a \to 1 \) for every finite \( F \subset Z^* \). In particular \( \Lambda(a) \to \infty \) when \( a \to \infty \) (i.e. when \( a_1, \ldots, a_v \to \infty \)).

Let \( Z^* \) act by homeomorphisms on the metrizable compact set \( \Omega \), and let \( d \) be a metric on \( \Omega \). \( C(\Omega) \) is the Banach space of real continuous functions on \( \Omega \) with the sup norm, and \( C(\Omega)^* \) the space of real measures on \( \Omega \) with the vague topology. The two assumptions below will be made throughout what follows.

2.1. Expansiveness. There exists \( \delta^* > 0 \) such that

\[
(d(mx, my) \leq \delta^* \text{ for all } m \in Z^*) \Rightarrow (x = y).
\]

2.2. Specification. Given \( \delta > 0 \) there exists \( p(\delta) > 0 \) with the following property. If \((\Lambda_a)\) is a family of subsets of \( \Lambda(a) \) such that the sets \( \Lambda_i + Z^*(a) \)
have mutual (Euclidean) distances $p(\delta)$, and if $(x_i)$ is a family of points of $\Omega$, there exists $x \in \Pi(a)$ such that

$$d(mx_i, mx) < \delta$$

for all $m \in \Lambda_1$, all $i$.

If $\Omega$ is a basic set for an Axiom A diffeomorphism ($\nu = 1$), it is known that expansiveness [14] holds, and that specification [1] holds for some iterate of the diffeomorphism.

3. Pressure and entropy. Letting $\delta > 0$, we say that $E \subset \Omega$ is $(\delta, \Lambda)$-separated if

$$(x, y) \in E \text{ and } d(mx, my) < \delta \text{ for all } m \in \Lambda \Rightarrow (x = y).$$

Let $\phi \in C(\Omega)$. Given $\delta > 0$ and a finite $\Lambda \subset Z^*$, or given $a = (a_1, \ldots, a_s)$ we introduce the “partition functions”

$$Z(\phi, \delta, \Lambda) = \max_E \sum_{x \in E} \exp \sum_{m \in \Lambda} \phi(mx),$$

where the max is taken over all $(\delta, \Lambda)$ separated sets, or

$$Z(\phi, a) = \sum_{x \in \Pi(a)} \exp \sum_{m \in \Lambda(a)} \phi(mx).$$

3.1. Theorem. If $0 < \delta < \delta^*$, the following limits exist:

$$\lim_{\Lambda \to \infty} \frac{1}{\text{card } \Lambda} \log Z(\phi, \delta, \Lambda) = P(\phi),$$

$$\lim_{a \to \infty} \frac{1}{\text{card } \Lambda(a)} \log Z(\phi, a) = P(\phi),$$

where $P$ defines a real convex function on $C(\Omega)$ such that

$$|P(\phi) - P(\psi)| \leq \|\phi - \psi\|;$$

$P$ is called the pressure.

Other definitions of $P$, using open coverings or Borel partitions of $\Omega$, are possible.

Let $\mathcal{A} = (A_j)_{j \in J}$ be a finite Borel partition of $\Omega$, and $\Lambda$ a finite subset of $Z^*$. We denote by $\mathcal{A}^{\Lambda}$ the partition of $\Omega$ consisting of the sets $A(k) = \bigcap_{m \in \Lambda} (-m)A_{k(m)}$ indexed by maps $k: \Lambda \to J$. We write

$$S(\mu, \mathcal{A}) = -\sum_j \mu(A_j) \log \mu(A_j).$$

Let $I$ be the (convex compact) set of $Z^*$ invariant probability measures on $\Omega$.

3.2. Theorem. If $\mathcal{A}$ consists of sets with diameter $\leq \delta^*$ and $\mu \in I$, then
This limit is finite $\geq 0$, and independent of $\mathcal{A}$. Furthermore, $s$ is affine upper semi-continuous on $I$; $s$ is called the entropy.

For $\nu = 1$, this is the usual definition of the measure theoretic entropy. Specification is not used in the proof of Theorem 3.2.

4. Variational principle and equilibrium states. Let $I$ be the set of $\mu \in C(\Omega)^*$ such that
\[
\forall \psi \in C(\Omega), \quad P(\phi + \psi) \supseteq P(\phi) + \mu(\psi) \quad \forall \psi \in C(\Omega).
\]
Those $\mu$ are called equilibrium states for $\phi$.

4.1. Theorem. The following variational principle holds:
\[
P(\phi) = \max_{\mu \in I} [s(\mu) + \mu(\phi)].
\]
The maximum is reached precisely for $\mu \in I_{\phi}$ (in particular $I_{\phi} \subset I$). The set $I_{\phi}$ is not empty; it is a Choquet simplex, and a face of $I$ [3]. There is a residual subset $D$ of $C(\Omega)$ such that $I_{\phi}$ consists of a single point $\mu_{\phi}$ if $\phi \in D$. For all $\mu \in I$,
\[
s(\mu) = \inf_{\phi \in C(\Omega)} [P(\phi) - \mu(\phi)].
\]

If $\Omega$ is a basic set for an Axiom A diffeomorphism it is known [2] that $0 \in D$, and (*) for $\phi = 0$ is related to the fact that the topological entropy is the sup of the measure theoretic entropy [4], [6]. Further results on $D$ have been obtained for Anosov diffeomorphisms using methods of statistical mechanics [13].

4.2. Theorem. Let $\mu_{\phi,a}$ be the measure on $\Omega$ which is carried by $\Pi(a)$ and gives $x \in \Pi(a)$ the mass
\[
\mu_{\phi,a}(\{x\}) = Z(\phi, a)^{-1} \exp \sum_{m \in \Lambda(a)} \phi(mx).
\]
If $\mu$ is a (vague) limit point of the $(\mu_{\phi,a})$ when $a \to \infty$, then $\mu \in I_{\phi}$. In particular, if $\phi \in D$,
\[
\lim_{a \to \infty} \mu_{\phi,a} = \mu_{\phi}.
\]

REFERENCES


