SINGULAR INTEGRALS IN THE SPACES $\Lambda(B, X)$

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In this note we describe the theory of singular integral and multiplier transformations in the setting of the spaces $\Lambda(B, X)$. First we introduce the spaces $\Lambda(B, X)$ combining Theorems A and B below, essentially due to A. P. Calderon, with paragraphs 14 and 34 of [2]. Theorem C and the example that follows it illustrate the fact that $\Lambda(B, X)$ spaces are related to Lipschitz spaces of functions and distributions in $R^n$. (See also [5].)

Then guided by the translation invariance of an important class of singular integrals, described before Theorem D, we define a class of operators which commute with representations of $R^n$ into a group of (uniformly) bounded linear operators of a Banach space $B$ into itself. The continuity of these singular integral operators is proved in Theorem D. Theorems E and F concerning multipliers are then proved with the assumption that the representations alluded to above are the translations. These results were submitted as a thesis at the University of Chicago.

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The spaces $\Lambda(B, X)$. Let $\{t^P\}_{t>0}$ be a group of transformations of $R^n$ where $P$ is a real $n \times n$ matrix and

$$(*) \quad \|t^P\| \leq t, \quad 0 < t \leq 1.$$ 

This will ensure the existence of a unique value $s$ for which $s^{-P}x \in S^{n-1}, x \neq 0$. Thus setting $\rho(x) = s$ we have that $\rho(t^P x) = t \rho(x)$ and $\rho(x + y) \leq \rho(x) + \rho(y)$. (See [4].) We notice that $(P x, x) \geq (x, x)$ is a necessary and sufficient condition for $(*)$ to hold. Moreover the adjoint matrix $P^*$ also satisfies $(P^* x, x) \geq (x, x)$ and therefore it determines a function $\rho^*(x)$ with similar properties. Now we construct a one-parameter family of dilations $v_t$ of a finite Borel measure $v$ on $R^n$ by setting $v_t(E) = v(t^{-P} E)$ for every $v$-measurable set $E$ and $t > 0$. If $dv(x) = \phi(x) \, dx$ where $\phi \in L^1(R^n)$, then $dv_t(x) = t^{-nP} \phi(t^{-P} x) \, dx$.

Let $B$ be a Banach space of tempered distributions on $R^n$ such that $\mathcal{S}(R^n) \subset B$ and $B = V^*$ for some complex Banach space $V$. For $y \in R^n$...
let $\tau_y$ be a representation of $R^n$ into a group of (uniformly) bounded linear operators of $B$ into itself, i.e. $\|\tau_y u\|_B \leq c\|u\|_B$, such that, for each $u \in B$ and $v \in V$, $(\tau_y u, v)$ is a continuous function of $y$. We now define \( \int \tau_y u \, dv(y) = (\tau_y u, v) \) for all $v \in V$. Analogously we define $F(y, t)$ acting on $v \in V$ as $(F(y, t), v) = \int (\tau_{y+t} u, v) \, dv(z)$.

**Theorem A.** Let $(\tau_y u, v)$ be a continuous function of $y \in R^n$ for $u \in B$ and $v \in V$. Let $v$ satisfy (i) $\|x^M|v| \|_X \leq C_M < \infty$ for all multi-indices $M$ of nonnegative integers and (ii) $(v, \psi) = \hat{\phi}(t^p \cdot x) \neq 0$ as a function of $t > 0$ for all $x \in R^n - (0)$. Then there exist functions $\phi, \psi \in \mathcal{S}(R^n)$ such that

$$ (u, v) = \int (\tau_y u, v) \phi(y) \, dy + \lim_{t \to 0} \int (F(y, t), v) \psi(y) \, dy \frac{dt}{t}. $$

In fact these functions may be chosen so that $\hat{\psi} \in C_0(R^n)$ vanishes in a neighbourhood of the origin and $\hat{\phi} \in C_0(R^n)$.

We would like to construct a similar representation for elements in $\Lambda(B, X)$. In order to do so we introduce some definitions.

A lattice $X$ of locally integrable functions in $(0, 1)$ is a linear class of functions such that there is a norm defined on $X$ with respect to which it is complete and if $f \in X$ and $|g| \leq |f|$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Given a positive monotone increasing (in the wide sense) submultiplicative function $p(t)$ defined on $(0, \infty)$ we say that $X$ is a $p$-lattice if the mappings

$$ f \to \int_0^t f(s) p(t/s)(t/s)^r \, ds$$

are continuous from $X$ into itself for $r > 0$. We choose to call a $t^r$-lattice an $r$-lattice. We set $\gamma X = \{ f \in L^1_{loc}(0, 1) : \gamma(t) f(t) E X \}$, $\|f\|_{\gamma X} = \|\gamma^{-1} f\|_X$.

Given $B$ and $X$ we denote by $X(B) = \{ F : F$ is $B$-valued weakly measurable and $\|F\|_B \in X \}$, $\|F\|_{X(B)} = \|\|F\|_B\|_X$. Finally we let $\Lambda(B, X) = \{ u \in B : \int \tau_y u \, dv(y) \in X(B) \}$. Normed with $\|u\|_{\Lambda_u} = \|u\|_B + \|\int \tau_y u \, dv(y)\|_{X(B)}$, $\Lambda(B, X)$ becomes a Banach space in which $B$ is continuously embedded.

Let $A_k$ be the class of $L^1(R^n)$ functions $f(x)$ such that there exists a “polynomial” $\sum |M| < k a_M(x) y^M$ so that

(i) $a_M(x) \in L^1$ and

(ii) $\int |f(x - y) - \sum a_M(x) y^M| \, dx = O(|y|^k)$ for $y \in R^n$.

**Theorem B.** Let $k \in Z^+$ and let $\mu$ be a Borel measure such that (i) $\int |x^M| \, d\mu(x) = 0$ for $|M| < k$; (ii) $\int |x|^k \, d\mu(x) < \infty$. Further let $X$ be a $\beta$-lattice and let $\gamma(t)$ be such that $\beta(t) \gamma(t)/t^k$ increases for some $\varepsilon > 0$ and $\beta(t) \gamma(t)/t^{k-\delta}$ decreases for some $k > \delta > 0$. Then for fixed functions $\phi, \psi \in A_k$ and
elements \( u \in B, F(t) \in X(B) \) the integrals

\[
\mathcal{S}(u, F) = \int \tau_x u \phi(y) \, dy + \int_0^1 \tau_x F(t) \psi(t) \gamma(t) \, dt
\]

converge absolutely in the \( B \)-norm and

\[
\| \mathcal{S}(u, F) \|_{\Lambda(B, X)} \leq C \{ \| u \|_B + \| F \|_{X(B)} \},
\]

\( C \) independent of \( u, F \).

**Corollary B.** If the hypotheses above hold and \( \gamma(t) = 1 \), and \( \mu = \nu, \phi, \psi \) as in Theorem A, then \( \mathcal{S}(u, F) \) maps \( B \oplus X(B) \) onto \( \Lambda(B, X) \). Also if now the measures \( \mu \neq \nu \) satisfy both the hypotheses of Theorems A and B, then \( \Lambda(B, X) \) and \( \Lambda(B, X) \) coincide algebraically and topologically. This explains the denotation \( \Lambda(B, X) \) for these spaces from now on.

**Corollary B.** [Cf. [1, paragraph 6.1.]] Let \( \phi \in L^1(\mathbb{R}^n) \), supp \( \phi \) compact and let \( X \) be an \( r \)-lattice, \( 0 < r < 1 \). Then \( \| u - \int \tau_x u \phi(y) \, dy \|_B \in X \) implies \( \| u - \tau_x u \|_B \in X \) for any \( z \in \mathbb{R}^n \).

We now characterize the spaces \( \Lambda(B, X) \) when \( P = \text{diag}(a_1, \ldots, a_n) \), \( a_i \in \mathbb{Z}^+ \). (For \( P = \text{identity} \) see [3, paragraph 14.1].) Put \( a = (a_1, \ldots, a_n) \).

Let

\[
v_z(y) \equiv v(y) = \sum_{j=0}^{k} \binom{k}{j} \left(-1\right)^j \delta(y - j^p z),
\]

where \( \delta \) is the Dirac measure centered at the origin. Moreover, let

\[
\Delta_{t,z} u = \int \tau_x u \, dv_z(y) = \sum_{j=0}^{k} \binom{k}{j} \left(-1\right)^j \tau_{t,j} P_z u.
\]

We then have

**Theorem C.** Let \( P, v, \Delta_{t,z} \) be as above and let \( X \) be an \( r \)-lattice. Furthermore assume that \( (\tau_x u, v) = (u, \tau_{t,v}) \), i.e. the \( \tau_x \) are the adjoints of a family \( \tau_y \) acting on \( V \). This assumption will be kept for the remainder of the note. Then for multi-indices \( M \) satisfying \( 0 < r - (a, M) < k \) we have

\[
\Lambda(B, X) = \{ u \in B : (\partial/\partial x)^M \tau_x u \}_{x=0} \in B \text{ and } \sup_{|z|=1} \| t^{(a,M)} \Delta_{t,z} (\partial/\partial x)^M \tau_x u \|_{x=0} \in X \}.
\]

Moreover

\[
\| u \|_{\Lambda} \leq \sup_{|z|=1} \{ (\partial/\partial x)^N \tau_x u \}_{x=0} \|_{B}, \| t^{(a,M)} \Delta_{t,z} (\partial/\partial x)^M \tau_x u \|_{x=0} \|_{X(B)} \},
\]

the supremum being taken over \( z \in S^{n-1} \), \( (a, N) < \xi \), \( (a, M) < r \). For example, let \( B = L^p(\mathbb{R}^n) \), \( 1 < p \leq \infty \), put \( \tau_x u = u(\gamma^{-1}(a, y)) \) and \( X = X_{r,p} \cap X_{s,p} \).
where \( X_{r,p} \) is the \( r \)-lattice \( t' L'(0,1; dt/t) \). If \( a = (1, \ldots, 1, 2) \), \( z = (z_1, \ldots, z_n-1, 0, 0) \in S^{n-1} \), \( \hat{z} = (0, \ldots, 0, 1) \) we obtain \( \Lambda(B,X) = \{ u \in L^p(R^n) : (\partial/\partial x)M u \in L^p(R^n) \} \) for \( M = (M_1, \ldots, M_n) \) with \( M_i \leq 2 \) for \( 1 \leq i \leq n-1 \) and \( M_n \leq 1 \) and \( \| t^2 \Delta_{r,z}(\partial/\partial x)M u \|_{X_{r,p}(L^r)} + \| t^2 \Delta_{r,z}(\partial/\partial x)M u \|_{X_{r,p}(L^r)} < \infty \) for \( (a,M) = 2 \) and \( 2 < r, s < k + 2 \). This is a "parabolic" Lipschitz space of functions with the last variable distinguished.

**Singular integrals.** Let \( k \in \mathcal{S}'(R^n) \) be defined by \( k(\phi) = \text{p.v.}\int k(x)\phi(x) \, dx \) for \( \phi \in \mathcal{S}(R^n) \), where \( k(x) \in L^1_{\text{loc}}(R^n - (0)) \) and it satisfies

(i) for \( 0 < r < R, \int_{\rho(x)<R} \rho(x) k(x) \, dx \leq C \) and \( \int_{\rho(x)<1} k(x) \, dx \) converges as \( r \to 0 \);

(ii) for \( R > 0, \int_{\rho(x)<R} \rho(x) k(x) \, dx \leq CR \) and

(iii) \( \int_{\rho(x)>r} |k(x - y) - k(x)| \, dx \leq C. \) [See [4] and [6].]

We define for \( u \in B \) the singular integral \( Ku \) as

\[
(Ku, v) = \lim_{\varepsilon \to 0} \int_{\varepsilon < x < 1/\varepsilon} (\tau_{\varepsilon} u, v)(y) \, dy, \quad \text{for all } v \in V.
\]

The next theorem is better understood if we recall the following remark due to Taibleson [7, p. 828]. The Riesz transforms \( R_i \) defined by \( (R_i u)(x) = x_i/|x|u(x) \) are not (in the notation of [7]) bounded mappings of \( \Lambda(x, p, \infty) \) into itself for \( p = 1, \infty \).

**Theorem D.** Let \( A = U^*, B = V^*, C = W^* \) be Banach spaces of tempered distributions such that \( K : A \to B \) continuously. Further assume that \( \mathcal{S} \) is dense in \( V \cap W \) and that if \( \phi(x) \) is the function of Theorem A then \( \int \tau_{\varepsilon} M \phi(y) \, dy : B \to C \) continuously. Then

(i) \( \tau_z K = K \tau_z \) and \( \tau_z Ku, v \) is a bounded function of \( z \in R^n \) for \( u \in A \) and \( v \in V \).

(ii) \( K \) is a continuous mapping from \( A \cap \Lambda(C, X) \) into \( B \cap \Lambda(C, X) \).

The proof uses Theorems A and B.

**Multipliers.** In this section we assume that the \( \tau_z \) act on \( \mathcal{S}'(R^n) \) by \( \tau_z \phi = (u, \phi \cdot - y) \) for all \( \phi \in \mathcal{S}(R^n) \). A function \( m(x) \) continuous and bounded in \( R^n - (0) \) is said to be a multiplier of type \( (A,B) \), where \( \mathcal{S} \) is dense in \( A \), if the mapping \( M \) defined by \( (Mu)(x) = m(x)u(x), u \in \mathcal{S} \), satisfies \( \| Mu \|_B \leq C \| u \|_A \). Clearly \( M \tau_z = \tau_z M \).

**Theorem E.** Let \( A = U^*, B = V^* \subset \mathcal{S}'(R^n) \). Let \( M : U \to V \) be such that \( M^* \tau_y = \tau_y M^*, M^* : A \to B \) continuously. If \( \tau_y \) maps boundedly \( A \) into itself and \( B \) into itself, then \( M^* : \Lambda(A, X) \to \Lambda(B, X) \) continuously.

The following is a particular instance of a more general valid fact.

**Theorem F.** Let \( A, B \) be as above and let \( C = W^* \subset \mathcal{S}'(R^n) \) be such that \( \int \tau_y \phi(y) \, dy : B \to C \), where \( \phi(y) \) is as in Theorem A. Let \( \mathcal{S}'(R^n) \) be dense
in $A \cap C$ and $V \cap W$. If $m(x)$ is $[n/2] + 1$ continuously differentiable in $R^n - (0)$ and if for $\Omega = \{0 < \varepsilon < \rho^*(x) < \varepsilon^{-1}\}$, where the choice of $\varepsilon$ depends solely on the function $\psi(x)$ of Theorem A, we have that

$$\sum_{|M| < [n/2] + 1} \int_{\Omega} |(\partial/\partial z)^M m(t^{n^*} z)|^2 \, dz \leq C \quad \text{for} \ t \geq 1,$$

then $m(x)$ is a multiplier of type $(A \cap \Lambda(C, X), B \cap \Lambda(C, X))$.

REFERENCES


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