BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 6, November 1972

SINGULAR INTEGRALS IN THE SPACES $\Lambda(B, X)$

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Communicated by A. Calderón, April 17, 1972

In this note we describe the theory of singular integral and multiplier transformations in the setting of the spaces $\Lambda(B, X)$. First we introduce the spaces $\Lambda(B, X)$ combining Theorems A and B below, essentially due to A. P. Calderon, with paragraphs 14 and 34 of [2]. Theorem C and the example that follows it illustrate the fact that $\Lambda(B, X)$ spaces are related to Lipschitz spaces of functions and distributions in R^n. (See also [5].) Then guided by the translation invariance of an important class of singular integrals, described before Theorem D, we define a class of operators which commute with representations of R^n into a group of (uniformly) bounded linear operators of a Banach space B into itself. The continuity of these *singular integral operators* is proved in Theorem D. Theorems E and F concerning multipliers are then proved with the assumption that the representations alluded to above are the translations. These results were submitted as a thesis at the University of Chicago. I would like to thank Professor A. P. Calderón for having had the privilege of learning with him these and many other things and to Professor Max Jodeit, Jr. for his help throughout my graduate studies.

The spaces $\Lambda(B, X)$. Let $\{t^P\}_{t>0}$ be a group of transformations of R^n where P is a real $n \times n$ matrix and

$$(*) \qquad \|t^P\| \leqq t, \qquad 0 < t \leqq 1.$$

This will ensure the existence of a unique value s for which $s^{-P}x \in S^{n-1}$, $x \neq 0$. Thus setting $\rho(x) = s$ we have that $\rho(t^P x) = t\rho(x)$ and $\rho(x + y) \leqq \rho(x) + \rho(y)$. (See [4].) We notice that $(Px, x) \geqq (x, x)$ is a necessary and sufficient condition for $(*)$ to hold. Moreover the adjoint matrix P^* also satisfies $(P^*x, x) \geqq (x, x)$ and therefore it determines a function $\rho^*(x)$ with similar properties. Now we construct a *one-parameter family of dilations* v_t of a finite Borel measure v on R^n by setting $v_t(E) = v(t^{-P}E)$ for every v-measurable set E and $t > 0$. If $dv(x) = \phi(x)\,dx$ where $\phi \in L^1(R^n)$, then $dv_t(x) = t^{-\operatorname{tr} P}\phi(t^{-P}x)\,dx$.

Let B be a Banach space of tempered distributions on R^n such that $\mathscr{S}(R^n) \subset B$ and $B = V^*$ for some complex Banach space V. For $y \in R^n$

AMS subject classifications. Primary 44A25, 26A16; Secondary 42A18, 26A69.
Key words and phrases. Spaces $\Lambda(B, X)$, singular integrals.
[1] This research was partially supported by the NSF Grant GP 23563.

let τ_y be a representation of R^n into a group of (uniformly) bounded linear operators of B into itself, i.e. $\|\tau_y u\|_B \leqq c\|u\|_B$, such that, for each $u \in B$ and $v \in V$, $(\tau_y u, v)$ is a continuous function of y. We now define $\int \tau_y u \, dv(y)$ to be the element $w \in B$ such that $(w, v) = \int (\tau_y u, v)\, dv(y)$ for all $v \in V$. Analogously we define $F(y, t)$ acting on $v \in V$ as $(F(y,t), v) = \int (\tau_{y+z} u, v)\, dv_t(z)$.

THEOREM A. *Let $(\tau_y u, v)$ be a continuous function of $y \in R^n$ for $u \in B$ and $v \in V$. Let v satisfy* (i) $\int |x^M|\,|dv(x)| \leqq C_M < \infty$ *for all multi-indices M of nonnegative integers and* (ii) $(v_t)^\wedge(x) = \hat{v}(t^{P^*}x) \not\equiv 0$ *as a function of $t > 0$ for any $x \in R^n - (0)$. Then there exist functions $\phi, \psi \in \mathscr{S}(R^n)$ such that*

$$(u, v) = \int (\tau_y u, v)\phi(y)\, dy + \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \int (F(y,t), v)\psi_t(y)\, dy \frac{dt}{t}.$$

In fact these functions may be chosen so that $\hat{\psi} \in C_0^\infty(R^n)$ vanishes in a neighbourhood of the origin and $\hat{\phi} \in C_0^\infty(R^n)$.

We would like to construct a similar representation for elements in $\Lambda(B, X)$. In order to do so we introduce some definitions.

A *lattice* X of locally integrable functions in $(0, 1)$ is a linear class of functions such that there is a norm defined on X with respect to which it is complete and if $f \in X$ and $|g| \leqq |f|$, then $g \in X$ and $\|g\|_X \leqq \|f\|_X$. Given a positive monotone increasing (in the wide sense) submultiplicative function $\beta(t)$ defined on $(0, \infty)$ we say that X is a *β-lattice* if the mappings

$$f \to \int_0^t f(s)\beta(t/s)(t/s)^{-\varepsilon}\frac{ds}{s} \quad \text{and} \quad f \to \int_t^1 f(s)\beta(t/s)(t/s)^{\varepsilon}\frac{ds}{s}$$

are continuous from X into itself for $\varepsilon > 0$. We choose to call a t^r-lattice an *r-lattice.* We set $\gamma X = \{f \in L^1_{\mathrm{loc}}(0,1) : \gamma(t)^{-1}f(t) \in X\}$, $\|f\|_{\gamma X} = \|\gamma^{-1}f\|_X$.

Given B and X we denote by $X(B) = \{F : F$ is B-valued weakly measurable and $\|F\|_B \in X\}$, $\|F\|_{X(B)} = \|(\|F\|_B)\|_X$. Finally we let $\Lambda_v(B, X) = \{u \in B : \int \tau_y u \, dv_t(y) \in X(B)\}$. Normed with $\|u\|_{\Lambda_v} = \|u\|_B + \|\int \tau_y u\, dv_t(y)\|_{X(B)}$, $\Lambda(B, X)$ becomes a Banach space in which B is continuously embedded.

Let A_k be the class of $L^1(R^n)$ functions $f(x)$ such that there exists a "polynomial" $\sum_{|M|<k} a_M(x)y^M$ so that

(i) $a_M(x) \in L^1$ and

(ii) $\int |f(x-y) - \sum a_M(x)y^M|\, dx = O(|y|^k)$ for $y \in R^n$.

THEOREM B. *Let $k \in Z^+$ and let μ be a Borel measure such that* (i) $\int x^M\, d\mu_t(x) = 0$ *for $|M| < k$;* (ii) $\int |x|^k\, |d\mu(x)| < \infty$. *Further let X be a β-lattice and let $\gamma(t)$ be such that $\beta(t)\gamma(t)/t^\varepsilon$ increases for some $\varepsilon > 0$ and $\beta(t)\gamma(t)/t^{k-\delta}$ decreases for some $k > \delta > 0$. Then for fixed functions ϕ, $\psi \in A_k$ and* *elements $u \in B$, $F(t) \in X(B)$ the integrals*

$$\mathscr{S}(u, F) = \int \tau_y u \phi(y)\, dy + \int_0^1 \int \tau_y F(t) \psi_t(y) \gamma(t)\, dy \frac{dt}{t}$$

converge absolutely in the B-norm and

$$\|\mathscr{S}(u, F)\|_{\Lambda_\mu(B, \gamma X)} \leqq C\{\|u\|_B + \|F\|_{X(B)}\},$$

C independent of u, F.

COROLLARY B_1. *If the hypotheses above hold and $\gamma(t) \equiv 1$, and $\mu = \nu$, ϕ, ψ are as in Theorem* A, *then $\mathscr{S}(u, F)$ maps $B \oplus X(B)$ onto $\Lambda_\nu(B, X)$. Also if now the measures $\mu \neq \nu$ satisfy both the hypotheses of Theorems* A *and* B, *then $\Lambda_\mu(B, X)$ and $\Lambda_\nu(B, X)$ coincide algebraically and topologically. This explains the denotation $\Lambda(B, X)$ for these spaces from now on.*

COROLLARY B_2. [*Cf.* [**1**, *paragraph* 6.1.]] *Let* $\phi \in L^1(R^n)$, supp ϕ *compact and let X be an r-lattice,* $0 < r < 1$. *Then* $\|u - \int \tau_y u \phi_t(y)\, dy\|_B \in X$ *implies* $\|u - \tau_t P_z u\|_B \in X$ *for any* $z \in R^n$.

We now characterize the spaces $\Lambda(B, X)$ when $P = \operatorname{diag}(a_1, \ldots, a_n)$, $a_i \in Z^+$. (For $P =$ identity see [3, paragraph 14.1].) Put $a = (a_1, \ldots, a_n)$. Let

$$\nu_z(y) \equiv \nu(y) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \delta(y - j^P z),$$

where δ is the Dirac measure centered at the origin. Moreover, let

$$\Delta_{t,z} u = \int \tau_y u\, d\nu_t(y) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \tau_{(jt)} P_z u.$$

We then have

THEOREM C. *Let P, ν, $\Delta_{t,z}$ be as above and let X be an r-lattice. Furthermore assume that $(\tau_y u, v) = (u, \tilde{\tau}_y v)$, i.e. the τ_y are the adjoints of a family $\tilde{\tau}_y$ acting on V. This assumption will be kept for the remainder of the note. Then for multi-indices M satisfying $0 < r - (a, M) < k$ we have*

$$\Lambda(B, X) = \{u \in B : (\partial/\partial x)^M \tau_x u]_{x=0} \in B \text{ and } \sup_{|z|=1} \|t^{(a,M)} \Delta_{t,z} (\partial/\partial x)^M \tau_x u]_{x=0}\|_B \in X\}.$$

Moreover

$$\|u\|_\Lambda \cong \sup\{\|(\partial/\partial x)^N \tau_x u]_{x=0}\|_B, \|t^{(a,M)} \Delta_{t,z} (\partial/\partial x)^M \tau_x u]_{x=0}\|_{X(B)}\},$$

the supremum being taken over $z \in S^{n-1}$, $(a, N) < r$, $(a, M) < r$. For example, let $B = L^p(R^n)$, $1 < p \leqq \infty$, put $\tau_y u = u(x - y)$ and $X = X_{r,p} \cap X_{s,p}$

where $X_{r,p}$ *is the r-lattice* $t^r L^p(0,1; dt/t)$. *If* $a = (1, \ldots, 1, 2)$, $z = (z_1, \ldots, z_{n-1}, 0) \in S^{n-1}$, $\tilde{z} = (0, \ldots, 0, 1)$ *we obtain* $\Lambda(B, X) = \{u \in L^p(R^n) : (\partial/\partial x)^M u \in L^p(R^n)$ *for* $M = (M_1, \ldots, M_n)$ *with* $M_i \leqq 2$ *for* $1 \leqq i \leqq n-1$ *and* $M_n \leqq 1$ *and* $\|t^2 \Delta_{t,z}(\partial/\partial x)^M u\|_{X_{r,p}(L^p)} + \|t^2 \Delta_{t,\tilde{z}}(\partial/\partial x)^M u\|_{X_{s,p}(L^p)} < \infty$ *for* $(a, M) = 2$ *and* $2 < r, s < k+2\}$. *This is a "parabolic" Lipschitz space of functions with the last variable distinguished.*

Singular integrals. Let $k \in \mathscr{S}'(R^n)$ be defined by $k(\phi) = \text{p.v.}\int k(x)\phi(x)\, dx$ for $\phi \in \mathscr{S}(R^n)$, where $k(x) \in L^1_{\text{loc}}(R^n - (0))$ and it satisfies

(i) for $0 < r < R$, $\left|\int_{r<\rho(x)<R} \rho(x) k(x)\, dx\right| \leqq C$ and $\int_{r<\rho(x)<1} k(x)\, dx$ converges as $r \to 0$;

(ii) for $R > 0, \int_{\rho(x)<R} \rho(x)|k(x)|\, dx \leqq CR$ and

(iii) $\int_{\rho(x)>4\rho(y)} |k(x-y) - k(x)|\, dx \leqq C$. [See [4] and [6].]

We define for $u \in B$ the *singular integral* Ku as

$$(Ku, v) = \lim_{\varepsilon \to 0} \int_{\varepsilon < \rho(y) < 1/\varepsilon} (\tau_y u, v) k(y)\, dy, \quad \text{for all } v \in V.$$

The next theorem is better understood if we recall the following remark due to Taibleson [7, p. 828]. The Riesz transforms R_i defined by $(R_i u)\hat{}(x) = x_i/|x|\hat{u}(x)$ are not (in the notation of [7]) bounded mappings of $\Lambda(\alpha, p, \infty)$ into itself for $p = 1, \infty$.

THEOREM D. *Let* $A = U^*$, $B = V^*$, $C = W^*$ *be Banach spaces of tempered distributions such that* $K: A \to B$ *continuously. Further assume that* $\mathscr{S}$ *is dense in* $V \cap W$ *and that if* $\phi(x)$ *is the function of Theorem* A *then* $\int \tau_y u \phi(y)\, dy : B \to C$ *continuously. Then*

(i) $\tau_z K = K \tau_z$ *and* $(\tau_z K u, v)$ *is a bounded function of* $z \in R^n$ *for* $u \in A$ *and* $v \in V$.

(ii) K *is a continuous mapping from* $A \cap \Lambda(C, X)$ *into* $B \cap \Lambda(C, X)$.

The proof uses Theorems A and B.

Multipliers. In this section we assume that the τ_y act on $\mathscr{S}'(R^n)$ by $(\tau_y u, \phi) = (u, \phi(\cdot - y))$ for all $\phi \in \mathscr{S}(R^n)$. A function $m(x)$ continuous and bounded in $R^n - (0)$ is said to be a *multiplier* of type (A, B), where $\mathscr{S}$ is dense in A, if the mapping M defined by $(Mu)\hat{}(x) = m(x)\hat{u}(x)$, $u \in \mathscr{S}$, satisfies $\|Mu\|_B \leqq C\|u\|_A$. Clearly $M\tau_y = \tau_y M$.

THEOREM E. *Let* $A = U^*$, $B = V^* \subset \mathscr{S}'(R^n)$. *Let* $M: U \to V$ *be such that* $M^*\tau_y = \tau_y M^*$, $M^*: A \to B$ *continuously. If* τ_y *maps boundedly* A *into itself and* B *into itself, then* $M^*: \Lambda(A, X) \to \Lambda(B, X)$ *continuously.*

The following is a particular instance of a more general valid fact.

THEOREM F. *Let* A, B *be as above and let* $C = W^* \subset \mathscr{S}'(R^n)$ *be such that* $\int \tau_y u \phi(y)\, dy : B \to C$, *where* $\phi(y)$ *is as in Theorem* A. *Let* $\mathscr{S}(R^n)$ *be dense* *in* $A \cap C$ *and* $V \cap W$. *If* $m(x)$ *is* $[n/2] + 1$ *continuously differentiable in* $R^n - (0)$ *and if for* $\Omega = \{0 < \varepsilon < \rho^*(x) < \varepsilon^{-1}\}$, *where the choice of* ε *depends solely on the function* $\psi(x)$ *of Theorem* A, *we have that*

$$\sum_{|M| < [n/2]+1} \int_\Omega |(\partial/\partial z)^M m(t^{P^*} z)|^2 \, dz \leqq C \quad \textit{for } t \geqq 1,$$

then $m(x)$ *is a multiplier of type* $(A \cap \Lambda(C, X), B \cap \Lambda(C, X))$.

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