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GROUP ACTIONS ON POINCARÉ DUALITY SPACES

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Let $G = Z_p$ for p prime and $K = Z_p$, or let $G = S^1$ and $K = Q$, and let G act on the compact space X. In this paper, we outline two proofs of the following:

THEOREM. *Suppose the compact G-space X is a Poincaré duality space over K of formal dimension n. Then each connected component of the fixed point set is a Poincaré duality space over K, and, if* $G \neq Z_2$, *has formal dimension congruent to n* mod 2.

This solves affirmatively the conjecture of Su given in [**5**].

Let $E_G \to B_G$ be the universal bundle for G and let X_G be the balanced product $(X \times E_G)/G$. The basic tools for both proofs are the fibre space $X \to X_G \to B_G$ and the localization theorem of Borel ([**1**], [**4**]). In the case X is totally nonhomologous to zero in X_G, Bredon has proven the Su conjecture [**2**]. However, this condition can be replaced by the two lemmas below, and this constitutes our algebraic proof. The second proof involves applying the localization theorem to a Thom space.

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1. **Algebraic proof.** When $G = S^1$ or Z_2, $H^*(B_G) = K[t]$ where t is of degree two in the S^1 case and of degree one in the Z_2 case. If $G = Z_p$ for p odd, then $H^*(B_G) = K[t, s]/s^2 = 0$ where s has degree one and t degree two. We consider the cohomology spectral sequence of the fibre space $X \to X_G \to B_G$.

LEMMA 1. (1_r) *E_r is generated over $K[t]$ by $E_r^{0,*}$ and $E_r^{1,*}$ for $G \neq Z_2$ or S^1, and by $E_r^{0,*}$ for $G = Z_2$ or S^1.*

(2_r) *If $j \geqq r - 1$, cup product with t gives an isomorphism of $E_r^{j,k}$ into $E_r^{j+2,k}$ for $G \neq Z_2$ and of $E_r^{j,k}$ into $E_r^{j+1,k}$ for $G = Z_2$ $(r \geqq 2)$.*

LEMMA 2. *Suppose there is a fixed point. Then the fundamental class U of $H^n(X)$ survives in $E_\infty^{0,n}$ and if $u \in E_\infty^{0,*}$ is nontorsion with respect to $H^*(B_G)$, there exists a $v \in E_\infty^{0,*}$ such that $uv = U$ (cup product).*

Lemma 1 is proven by induction. 1_2 and 2_2 are true for $G = S^1$ since $E_2 = H^*(B_G) \otimes H^*(X)$, and for $G = Z_p$ by known results of homological

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algebra (see [3]). The induction step is then shown by straightforward degree arguments.

Lemma 2 is proven by restriction to a N-dimensional orientable submanifold $B \subset B_G$ for large N. Then since $H^n(X, Z_p) = Z_p$ and Z_p has no nontrivial action on Z_p, the local coefficients are trivial in the top dimension. Thus by piecing together over neighborhoods on which $X_G|_B$ is trivial, it is easy to show that $X_G|_B$ satisfies Poincaré duality with a cohomology fundamental class $[B]U$ where $X_G|_B$ is the portion of X_G over B and $[B]$ is the fundamental class of B. Using the fact that the inclusion $X_G|_B \to X_G$ induces an isomorphism on $E_2^{j,k}$ for $j \leqq N$ implies it induces an injection on $E_\infty^{j,k}$ for $j \leqq N$, and choosing N large enough so it induces an isomorphism on $E_\infty^{0,*}$, the lemma follows by finding a class in $H^*(X_G|_B)$ dual to $[B]u$.

With these two lemmas the proof of Bredon is valid without change.

2. **Geometric proof.** We shall assume that

(i) X can be embedded in Euclidean space as a neighborhood retract.

(ii) X has a finite number of orbit types.

Property (i) is inherited by the fixed point set, as follows from (ii) and the equivariant embedding theorem. Because of (i) X is a Poincaré duality space over K of formal dimension n if and only if for any embedding $X \subset S^{n+r} = S$ there is an isomorphism

$$x \mapsto x \cup U : H^*(X) \to H^*(S, S - X)$$

for some $U \in H^r(S, S - X)$.

Choose a G-equivariant embedding, and use K as the coefficient field. Then $H^r(S, S - X) = H^r_G(S, S - X)$ and we consider U as an element of both groups, where we define for any G-pair (A, A'), $H^*_G(A, A') = H^*(A_G, A'_G)$. By induction on the cells in B_G, we see that there is an isomorphism

$$\cup U : H^*_G(X) \to H^*_G(S, S - X).$$

Let Σ be the fixed sphere in S and $F = X \cap \Sigma$ the fixed set in X. Then the following diagram is commutative:

$$\begin{array}{ccc} H^*_G(S, S - X) & \xrightarrow{i^*} & H^*_G(\Sigma, \Sigma - F) \\ \cong \uparrow \cup U & & \uparrow \cup i^*(U) \\ H^*_G(X) & \xrightarrow{i^*} & H^*_G(F) \end{array}$$

After localizing, the maps i^* and hence all the maps in the diagram are isomorphisms. Localizing here means tensoring over $K[t] \subset H^*(B_G)$ with $K[t, t^{-1}]$. The map on the right splits according to the connected components of F, so we may assume F is connected. Then

$$i^*(U) = t^a u_0 + t^{a-1} u_1 + \dots + u_a + sv$$

where $u_i \in H^*(\Sigma, \Sigma - F)$, $u_0 \neq 0$, and $s = 0$ if $G = Z_2$ or S^1. Hence

$$\cup\, u_0 : H^*(F) \to H^*(\Sigma, \Sigma - F)$$

is an isomorphism, so F is a Poincaré duality space over K.

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