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FUNCTIONS OF SEVERAL NONCOMMUTING VARIABLES[1]

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The notions of nonsingularity, resolvent set, and spectrum, and the corresponding analytic functional calculus for n-tuples of elements of a commutative Banach algebra provide some of the deepest and most potent tools of modern analysis. A point of view one can adopt regarding this theory is as follows: The class of algebras $\mathcal{O}(U)$, for U a domain in C^n, provides a relatively small, well-understood, and nicely behaved class of topological algebras with distinguished tuples $(z_1, \ldots, z_n)$ of elements; furthermore, spectral theory and the Shilov-Arens-Calderón Theorem (cf. [**17**]) give precise information regarding which algebras $\mathcal{O}(U)$ can be mapped into a given commutative Banach algebra (or F-algebra) by a continuous homomorphism carrying $(z_1, \ldots, z_n)$ onto a specified tuple of elements. Thus, the algebras $\mathcal{O}(U)$ and tuples $(z_1, \ldots, z_n)$ provide a tractable class of models for the behavior of n-tuples of elements of a commutative topological algebra.

In [**7**] and [**8**] we showed how to extend spectral theory and the analytic functional calculus in a well-defined manner to the study of commuting n-tuples of operators on a Banach space. From this point of view, the pairs $(\mathcal{O}(U), (z_1, \ldots, z_n))$ provide models for the behavior of n-tuples of operators.

It must occur to nearly every analyst who encounters joint spectral theory to wonder whether or not there are useful notions of nonsingularity and spectrum for tuples in a noncommutative algebra or for noncommuting tuples of operators. From the point of view we have adopted regarding spectral theory, a more meaningful question is the following: Is there a reasonably small, well behaved class of pairs $(A, (z_1, \ldots, z_n))$, consisting of an algebra A and an n-tuple $(z_1, \ldots, z_n)$ of elements of A, that will serve as models for the behavior of fairly general (noncommutative) n-tuples of algebra elements or n-tuples of operators? Given such a class, the analogue of spectral theory and the functional calculus would consist of techniques for deciding which models $(A, (z_1, \ldots, z_n))$ can be mapped into a given algebra n-tuple pair $(B, (a_1, \ldots, a_n))$, or equivalently, which models

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$(A, (z_1, \ldots, z_n))$ have representations on a given space X for which the elements $z_1, \ldots, z_n$ act as given operators $a_1, \ldots, a_n \in L(X)$.

The above question has intrigued us for some time. Our definition in [7] of joint spectrum for commuting n-tuples of operators can be phrased in terms of concepts from homological algebra which make perfectly good sense in noncommutative situations (cf. [**9**]); that is, our version of commutative joint spectral theory can be viewed as the study of certain homological relations between modules over the n-variable polynomial algebra P_n; and these are relations which also make sense for modules over more general (even noncommutative) algebras. This suggested that homological algebra might provide the key to defining and studying a class of models for the noncommutative situation. The pursuit of this idea led to a paper [**10**] on homology and cohomology for topological algebras, and another [**11**] applying this work to obtain a version of spectral theory which makes sense in a wide variety of situations (both commutative and noncommutative). Here we shall review those portions of this latter paper (with the homological algebra suppressed) that relate directly to the study of noncommutative n-tuples of operators, and also present some more recent results on the subject.

In order to develop a class of models $(A, (z_1, \ldots, z_n))$ for the behavior of noncommutative n-tuples, we begin with the free algebra F_n on n-generators $z_1, \ldots, z_n$ and identify certain embeddings $F_n \to A$ of F_n in "satellite" algebras A. These algebras are to be in some sense algebras of functions of the free variables $z_1, \ldots, z_n$—just as the algebras $\mathcal{O}(U)$ are algebras of functions of the generators of the polynomial algebra P_n.

A brief discussion of the single variable functional calculus may be helpful in understanding what we shall be doing. Thus, let $a \in L(X)$ be a bounded linear operator on the Banach space X. Then a determines an algebra homomorphism $p \to p(a)$ from the polynomial algebra P into the algebra $L(X)$, that is, a representation of P on X (or a P-module structure on X). Now the algebra P is canonically embedded in each of the algebras $\mathcal{O}(U)$ (the algebra of all holomorphic functions on U) for U a domain in $\mathbf{C}$. Hence, we can ask the question: For which algebras $\mathcal{O}(U)$ does the representation of P on X determined by a extend to a (continuous) representation of $\mathcal{O}(U)$ on X? Of course, the answer is supplied by the analytic functional calculus: The representation extends to $\mathcal{O}(U)$ if and only if U contains the spectrum of the operator a; furthermore, the extension is unique if it exists.

The fact that the above question has such a nice answer is no accident. It is due to the fact that there is a very special relationship between the algebra P and the algebras $\mathcal{O}(U)$. If A is any topological algebra and φ a representation of A on an l.c.s. X, one could ask the question: For which embeddings, $A \to B$, of A in other topological algebras, B, is it true that the representation extends uniquely to a representation of B? One would not, however, expect such a general question to have a reasonable answer. For a given A, however, it might be possible to identify a class of "satellite" algebras B containing A for which the question is tractable for a sufficiently restricted class of representations φ. This is certainly the case for P; the class of "satellite" algebras is the class of algebras $\mathcal{O}(U)$; for these, and the representations of P on Banach spaces, the functional calculus problem has a nice answer.

What is special about the relationship between P and the algebras $\mathcal{O}(U)$? In some sense the elements of $\mathcal{O}(U)$ are functions of the generator of P. However, there is a more precise statement of this relationship, in homological terms, which can be generalized to other situations. Roughly speaking, an algebra embedding $A \to B$ is a localization if representations of A which extend to representations of B do so uniquely, and if the homological relations between representations of B (B-modules) are independent of whether the modules are considered A-modules or B-modules (cf. [**11**]). Here we are referring to a version of homological algebra developed for topological algebras (cf. [**10**]).

The idea, then, is to begin with a given "base" algebra A and study its family of "satellite" algebras B, i.e., those algebras which arise from localizations $A \to B$ of A. Representations of A are then classified according to which localizations $A \to B$ have the property that the representation extends to B.

For the base algebra P_n (the n-variable polynomial algebra), each of the natural embeddings $P_n \to \mathcal{O}(U)$, for U a domain of holomorphy in C^n, is a localization (cf. [**11**, §6]). However, these are not the only localizations of P_n; others include the natural map $P_n \to C^\infty(V)$, for V a domain in R^n, the map $P_n \to \mathcal{E}'(R^n)$ of P_n into the algebra of compactly supported distributions determined by sending the generators of P_n to the partial derivatives of the Dirac δ, and embeddings of P_n in algebras of rational functions.

Other base algebras whose localizations are discussed in [**11**] are the free algebra F_n on n-generators, the universal algebra of a Lie algebra, and the algebra of compactly supported distributions on a Lie group.

Every n-tuple of operators determines a representation of the free algebra F_n on n-generators. Thus, it is the natural base algebra to use in the study of noncommuting n-tuples. The condition that an algebra homomorphism $F_n \to A$ be a localization is equivalent to the existence of certain operators $\Delta_1, \ldots, \Delta_n$ from A to $A \hat{\otimes} A$ that have properties reminiscent of partial derivatives (cf. [**11**, §6]). In this paper we shall restrict our attention to localizations of F_n. We shall use the existence of the operators $\Delta_1, \ldots, \Delta_n$ to define the concept (rather than using the homological definition of **[11]**), and to develop several of its implications. There are remarkable similarities between localizations of F_n and ordinary algebras of functions of one or more variables; we shall stress these similarities throughout the paper.

In §1 we introduce localizations of F_n and discuss their elementary properties. In §2 we begin our discussion of the analytic functional calculus problem, i.e., the problem of deciding for which localizations $F_n \to A$ a given representation of F_n extends to a representation of A. This discussion is continued later, in §7, where we introduce a concept related to the resolvent set of ordinary spectral theory and outline a method of using this concept to attack the functional calculus problem; this method actually supplies a new proof of the ordinary single variable analytic functional calculus theorem.

The class of all localizations of F_n will prove too large for our purposes. In fact, the localizations of $F_1 = P$ include many algebras other than the algebras $\mathcal{O}(U)$. Thus, in §3 we introduce a class of localizations of F_n which we call free analytic algebras. These are localizations which are F-algebras such that the operators $\Delta_1, \ldots, \Delta_n$ satisfy certain regularity properties relative to a sequence of seminorms defining the topology of the algebra. In one variable, the free analytic algebras are just the algebras $\mathcal{O}(U)$ for U a domain in $\mathbf{C}$.

An algebra homomorphism $F_n \to A$ is determined by specifying the images $z_1, \ldots, z_n$ of the generators of F_n. If $F_n \to A$ is a localization (or free analytic algebra) we call $z_1, \ldots, z_n$ its generators. For a localization $F_n \to A$ with generators $z_1, \ldots, z_n$, it turns out that each representation of A is determined by its values on $z_1, \ldots, z_n$. Thus, we will commonly denote a representation of A on X by $u \to u(\alpha): A \to L(X)$, where $\alpha = (a_1, \ldots, a_n)$ is the image of the tuple $(z_1, \ldots, z_n)$; the a_i's will be called the generators of the representation. Free analytic algebras A have a fairly surprising property: If $u \to u(\alpha): A \to L(X)$ is a representation of A on a Banach space X, then for all $\beta = (b_1, \ldots, b_n)$ in a neighborhood of $\alpha = (a_1, \ldots, a_n) \in \bigoplus^n L(X)$, there is a representation $u \to u(\beta)$ of A on X with generators $b_1, \ldots, b_n$; furthermore, $u \to u(\beta)$ varies analytically with β in this neighborhood (cf. §3). There are even analogues of Taylor's formula and power series expansions for free analytic algebras; these are developed in §4.

Examples of free analytic algebras can be obtained by completing F_n in the topology generated by a sequence of power series type norms. The resulting algebras are the usual free analogues of power series algebras (cf. §3). However, these are not the only free analytic algebras. In §4 we describe an analogue of the implicit function theorem which specifies that certain sets of relations in a free analytic algebra A generate ideals I such that A/I is a free analytic algebra on fewer generators. This allows one to construct free analytic algebras in which certain elements have inverses or certain equations have solutions.

We conjecture that free analytic algebras must be Montel spaces (bounded sets are precompact). This is certainly true in one variable. In §5 we present a discussion and partial result on the general problem.

We end this introduction with an apology: Although we feel that the ideas presented here are promising, it is too early to predict whether or not a significant theory will result from further development. We know very little about the class of all free analytic algebras. As yet, we have no significant applications of what we do know. The techniques of §6 for attacking the functional calculus problem might prove to be practical, for $n > 1$, only in cases where the ensuing results were trivial anyway. Still, we feel quite strongly that free analytic algebras are the "correct" free analogues of algebras of analytic functions, and that this justifies pursuing the matter for awhile.

1. **Localizations of F_n.** Throughout our discussions we shall be dealing with complete, Hausdorff, locally convex algebras over C, with identity, in which the multiplication is at least separately continuous. Rather than repeating these conditions ad infinitum, we shall refer to such an object simply as an algebra. A map $A \to B$ between two algebras will be called a homomorphism if it is an algebraic homomorphism and is continuous; if it maps the identity to the identity, it will be called unital.

The theory of topological tensor products is a central tool for our work (cf. [**2**], [**5**], [**6**], [**12**]). Given two locally convex topological vector spaces (l.c.s.'s) X and Y, we denote the completed inductive tensor product of X and Y by $X \hat{\otimes} Y$. This is a complete l.c.s. with a separately continuous bilinear map $(x, y) \to x \otimes y : X \times Y \to X \hat{\otimes} Y$ such that any separately continuous bilinear map $\varphi : X \times Y \to Z$, into a complete l.c.s. Z, determines a unique continuous linear map $\tilde{\varphi} : X \hat{\otimes} Y \to Z$ with $\tilde{\varphi}(x \otimes y) = \varphi(x, y)$. Thus, the conditions we have imposed on an algebra A ensure that the multiplication map $A \times A \to A$ determines a continuous linear map $\pi : A \hat{\otimes} A \to A$ with $\pi(u \otimes v) = uv$.

For F-spaces X and Y, separately continuous linear maps $X \times Y \to Z$ are jointly continuous and the inductive tensor product $X \hat{\otimes} Y$ agrees with the projective tensor product (characterized as above, but with jointly continuous bilinear maps) (cf. [**5**, III, §5]). Most of the algebras and modules we are interested in are F-spaces.

For an algebra A, a left A-module will be a complete l.c.s. X together with an associative, separately continuous bilinear map $(u, x) \to ux$: $A \times X \to X$ such that $1x = x$. This map induces a unique continuous linear map $A \hat{\otimes} X \to X$ mapping $u \otimes x$ to ux. If X is a left A-module, then the map $u \to \alpha(u)$ $(\alpha(u)x = ux)$ defines a unital algebra homomorphism α of A into $L(X)$, the algebra of continuous linear maps on X. This map is continuous if $L(X)$ is given the topology of simple convergence (strong operator topology). We call α the representation of A associated with the module. Sometimes we shall find it convenient to use the module terminology and notation, while other times it will be more convenient to use the representation point of view.

Right A-modules are defined in an analogous fashion. Here, the module operation $(x, u) \to xu : X \times A \to X$ corresponds to an antihomomorphism $\alpha : A \to L(X)$ with $\alpha(u)x = xu$. An A-bimodule is an l.c.s. with left and right A-module operations which commute with each other $(u(xv) = (ux)v)$.

If A is an algebra, then $A \hat{\otimes} A$ has both an A-bimodule structure and its own algebra structure. The bimodule operations we shall use are defined by $u(v \otimes w) = uv \otimes w$ and $(v \otimes w)u = v \otimes wu$. The algebra operation we use on $A \hat{\otimes} A$ will be denoted $(f, g) \to f * g$ and is defined by $(u \otimes v) * (u' \otimes v') = uu' \otimes v'v$ (multiplication is reversed in the second factor). Note that, with this notation, the module operations can be written as $uf = (u \otimes 1) * f$, $fu = (1 \otimes u) * f$.

We shall use the notation δu for the element $u \otimes 1 - 1 \otimes u \in A \hat{\otimes} A$ determined by $u \in A$.

DEFINITION 1.1. Let A be an algebra and $F_n \to A$ a unital homomorphism with $z_1, \ldots, z_n$ the images of the generators. We shall say $F_n \to A$ is a localization if there exist continuous linear maps $\Delta_i : A \to A \hat{\otimes} A$ (for $i = 1, \ldots, n$) satisfying

(a) $\Delta_i z_j = \delta_{ij}(1 \otimes 1), i = 1, \ldots, n;$

(b) $\Delta_i(uv) = u\Delta_i(v) + \Delta_i(u)v$ for $u, v \in A, i = 1, \ldots, n;$

(c) $\delta u = (\Delta_1 u) * \delta z_1 + \cdots + (\Delta_n u) * \delta z_n.$

The above conditions should remind one of similar conditions satisfied by the partial differentiation operators $\partial/\partial z_i$ for holomorphic functions of several variables $z_1, \ldots, z_n$. However, this analogy cannot be carried too far. In fact, the operators Δ_i are more closely related to difference quotients than actual derivatives.

As an example, we consider the algebra $\mathcal{O}(U)$ of holomorphic functions on a domain $U \subset \mathbf{C}$. The natural embedding $P \to \mathcal{O}(U)$ is a localization of $F_1 = P$. The algebra $\mathcal{O}(U) \hat{\otimes} \mathcal{O}(U)$ may be identified with the algebra $\mathcal{O}(U \times U)$ of holomorphic functions of $(z, w) \in U \times U$ (cf. [**12**, Theorem 51.6]). The bimodule operations $(u, f) \to uf$ and $(f, u) \to fu$ are then given by $(uf)(z, w) = u(z)f(z, w)$ and $(fu)(z, w) = u(w)f(z, w)$. Also, for $u \in \mathcal{O}(U)$ the element δu is given by $\delta u(z, w) = u(z) - u(w)$. Thus, it is apparent that (a), (b), and (c) of Definition 1.1 will be satisfied if we define $\Delta : \mathcal{O}(U) \to \mathcal{O}(U \times U)$ by

$$\Delta u(z, w) = (z - w)^{-1}(f(z) - f(w)).$$

Of course, it is crucial that the derivative $f'(z)$ provides a holomorphic extension of this difference quotient to all of $U \times U$. Thus, the existence of Δ is related to the existence of the derivative in $\mathcal{O}(U)$, but Δ itself is the difference quotient.

Throughout the remainder of this section we shall assume that $F_n \to A$ is a localization of F_n with generators $z_1, \ldots, z_n$.

If α is a representation of A on X, then we shall write $u(\alpha)$, rather than $\alpha(u)$, for the element in $L(X)$ corresponding to $u \in A$. The reasons for this will soon become apparent. If α and β are representations on X and Y and if $c \in L(Y, X)$, then the trilinear map $(u, v, x) \to u(\alpha)cu(\beta)x : A \times A \times X \to Y$ is separately continuous and, hence, induces a continuous linear map $\mathcal{N}_c : A \mathbin{\hat{\otimes}} A \mathbin{\hat{\otimes}} X \to Y$. It follows that each $f \in A \mathbin{\hat{\otimes}} A$ defines a linear map $c \to f(\alpha, \beta)[c] : L(X, Y) \to L(X, Y)$ by $f(\alpha, \beta)[c](x) = \mathcal{N}_c(f \otimes x)$. If $f = u \otimes v$ then $f(\alpha, \beta)[c]$ is just $u(\alpha)cv(\beta)$. If we denote $z_i(\alpha)$ by a_i and $z_i(\beta)$ by b_i, then $\delta z_i(\alpha, \beta)[c] = a_i c - c b_i$. Condition (c) of Definition 1.1 then implies that

$$u(\alpha)c - cu(\beta) = \Delta_1 u(\alpha, \beta)[a_1 c - c b_1] + \cdots + \Delta_n u(\alpha, \beta)[a_n c - c b_n] \tag{1.1}$$

for all $u \in A$.

Equation (1.1) has several consequences. In the case $X = Y$ and $c = 1 \in L(X)$, we conclude that two representations α and β of A on X agree if they agree on the generators ($a_i = b_i$). Hence, a representation α of A is uniquely determined by the tuple of operators $(a_1, \ldots, a_n)$ $(a_i = z_i(\alpha))$; for this reason, we will henceforth specify a representation by specifying the pair (X, α) with $\alpha = (a_1, \ldots, a_n)$. The operators $a_1, \ldots, a_n$ will be called the generators of the representation. The image of $u \in A$ under the representation will be denoted $u(\alpha)$.

A homomorphism $c: Y \to X$ between two representations (X, α) and (Y, β) is an element $c \in L(Y, X)$ such that $u(\alpha)c = cu(\beta)$ for all $u \in A$ (a left A-module homomorphism). It follows from (1.1) that c is a homomorphism if and only if $a_i c = c b_i$ for $i = 1, \ldots, n$, where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are the generators. For a single representation generated by $\alpha = (a_1, \ldots, a_n)$, this implies that each $u(\alpha)$ commutes with every operator in $L(X)$ that commutes with each a_i.

Now every tuple of operators $\alpha = (a_1, \ldots, a_n)$ on a complete l.c.s. X determines a unique representation of F_n in which the generators are mapped to $a_1, \ldots, a_n$. In particular, the elements $z_1, \ldots, z_n \in A$ determine the canonical embedding $F_n \to A$ of F_n in A. With this in mind, the preceding discussion may be summarized as follows:

PROPOSITION 1.1. *Let $F_n \to A$ be a localization of F_n. Then*

(a) *if* (X, α) *is a representation of* F_n, *there is at most one extension to a representation* $u \to u(\alpha)$ *of* A; *if such an extension exists, each* $u(\alpha)$ *commutes with every* $c \in L(X)$ *that commutes with the generators* $a_1, \ldots, a_n$;

(b) *given two* A*-modules* (X, α) *and* (Y, β) *and* $c \in L(Y, X)$, c *is an* A*-module homomorphism if and only if it is an* F_n*-module homomorphism.*

The properties expressed in Proposition 1.1 are not unique to localizations of F_n. In fact, we could draw the same conclusions if we simply assumed that for each $u \in A$ there were elements $f_1, \ldots, f_n \in A \hat{\otimes} A$ (not necessarily unique) for which

$$\delta u = f_1 * \delta z_1 + \cdots + f_n * \delta z_n. \tag{1.2}$$

This is a property possessed by all factor algebras of F_n or any of its localizations. Such an algebra would be called a pseudo-quotient of F_n in the terminology of [**11**]. The fact that, for a localization, the f_i's are given by continuous linear maps $\Delta_i : A \to A \hat{\otimes} A$ with properties (a) and (b) of Definition 1.1 yields an additional property that is quite strong.

Suppose that (X, α) and (Y, β) are representations of A. For each n-tuple $(c_1, \ldots, c_n)$ in $L(Y, X)$ we consider the representation of F_n on $X \oplus Y$ generated by the tuple $\gamma = (g_1, \ldots, g_n)$, where $g_i(x, y) = (a_i x + c_i y, b_i y)$. In other words, g_i is represented by the matrix

$$\begin{pmatrix} a_i & c_i \\ 0 & b_i \end{pmatrix}.$$

The maps Δ_i give us an explicit way of extending this representation to a representation of A on $X \oplus Y$. We simply let $u(\gamma)$, for $u \in A$, be the operator whose matrix is

$$\begin{pmatrix} u(\alpha) & \sum_i \Delta_i u(\alpha, \beta)[c_i] \\ 0 & u(\beta) \end{pmatrix}.$$

Then $u(\gamma)v(\gamma)$, for $u, v \in A$, is represented by

$$\begin{pmatrix} u(\alpha)v(\alpha) & u(\alpha)\sum_i \Delta_i v(\alpha, \beta)[c_i] + \sum_i \Delta_i u(\alpha, \beta)[c_i]v(\beta) \\ 0 & u(\beta)v(\beta) \end{pmatrix}.$$

But it follows from (b) of Definition 1.1 that

$$u(\alpha)\sum_i \Delta_i v(\alpha, \beta)[c_i] + \sum_i \Delta_i u(\alpha, \beta)[c_i]v(\beta) = \sum_i \Delta_i uv(\alpha, \beta)[c_i].$$

Hence, $u(\gamma)v(\gamma) = uv(\gamma)$ and $u \to u(\gamma)$ is a homomorphism. The continuity follows from the continuity of the Δ_i's. Hence, $u \to u(\gamma)$ is a representation of A. That it extends the given representation of F_n follows from (a) of Definition 1.1.

The representation of F_n on $X \oplus Y$ defined above has the property that the injection $X \to X \oplus Y$ and projection $X \oplus Y \to Y$ are homomorphisms. That is, (X, α) is a submodule of $(X \oplus Y, \gamma)$ and (Y, β) is a factor module. In general, if X and Y are modules over an algebra A, then an A-module structure on $X \oplus Y$ for which the canonical image of X in $X \oplus Y$ is a submodule and Y is the corresponding factor module is called an A-module extension of Y by X. Thus, we have proved that

PROPOSITION 1.2. *If $F_n \to A$ is a localization and X and Y are A-modules, every F_n-module extension of Y by X is also an A-module extension.*

We have already pointed out that the algebras $\mathcal{O}(U)$ are localizations of $F_1 = P$. Are there others? A moment's reflection should convince the reader that the algebra $C^\infty(I)$ (I an open interval on the line) is a localization with generator the identity function. Here again, the difference quotient provides the map Δ. It turns out that the algebra $\mathcal{E}'(R)$ of compactly supported distributions is a localization with the derivative of the Dirac delta as generator (cf. [**11**, §3]). Algebras of rational functions are also localizations (cf. [**11**, §3]).

Suppose A is any localization of P and let Ω be the set of those $\lambda \in C$ which generate one-dimensional representations $u \to u(\lambda)$ of A; that is, $\lambda \in \Omega$ if λ is the image of the generator z of A under some continuous complex homomorphism of A. Then each $u \in A$ determines a function $\lambda \to u(\lambda)$ on Ω and each $f \in A \hat{\otimes} A$ determines a function $(\lambda, \omega) \to f(\lambda, \omega)$ on $\Omega \times \Omega$. Let $\hat{A}$ denote the resulting algebra of functions on Ω. Note that, for $f \in A \hat{\otimes} A$, $f(\lambda, \omega)$ is in $\hat{A}$ as a function of $\lambda(\omega)$ for each fixed $\omega \in \Omega$ $(\lambda \in \Omega)$. Furthermore, we have

$$u(\lambda) - u(\omega) = (\lambda - \omega)\Delta u(\lambda, \omega)$$

from (1.1). It follows that if the elements of $\hat{A}$ are bounded on a subset $S \subset \Omega$, they are also continuous on S and differentiable in the sense that $(\lambda - \omega)^{-1}(u(\lambda) - u(\omega))$ converges as $\lambda \to \omega$ within S. In particular, each $\lambda \to u(\lambda)$ must be holomorphic on any open set S with this property. One way of insuring that there are large subsets of Ω on which functions in $\hat{A}$ are bounded is to demand that A be an F-algebra (cf. [**4**]). This insures that Ω is the union of a sequence of compact sets on which each $\lambda \to u(\lambda)$ is bounded.

The mapping $A \to \hat{A}$ defined above need not be one to one for a localization A of P. An example is the algebra of all convergent power series. Here Ω is the single point $\{0\}$.

There are probably too many pathological examples of localizations of P for a complete characterization to be practical. However, as we shall see in §3, by insisting that A be an F-algebra and that Δ satisfy a certain