BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

MANIFOLDS AS OPEN BOOKS

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Communicated by Michael Atiyah, June 5, 1972

Introduction. We prove that all closed, simply-connected, differentiable (or p.l.) manifolds of dimension >6 and index $\tau = 0$ decompose in a certain way: as "open books", a decomposition analogous to the classical Lefschetz decomposition of nonsingular algebraic varieties. The condition $\tau = 0$ is also necessary for a manifold to be an open book, and so, in particular, we have found a simple, intrinsic, geometric equivalent of it. For any orientable 3-manifold, this decomposition had been given by Alexander in 1923, using properties of branched coverings, which do not seem to generalize to higher dimensions. The proof of our theorem is not difficult and is a natural consequence of decomposing manifolds à la Heegaard, first accomplished for a large class of high-dimensional manifolds by Smale and completed by others [**2**], [**3**], [**8**].[2]

We wish to thank W. D. Neumann, J. W. Wood and especially Professor D. Sullivan of M.I.T. for very helpful conversations.

1. **Statement of the theorem.** Let V be a compact differentiable $(n-1)$-manifold with $\partial V \neq \emptyset$ and $h: V \to V$ a diffeomorphism, which restricts to the identity on ∂V; by forming the mapping torus V_h, which has $\partial V \times S^1$ as boundary, and identifying $(x, t) \sim (x, t')$ on ∂V_h for each $x \in \partial V$, $t, t' \in S^1$, we obtain a closed, differentiable n-manifold M, which, if we look at a piece of the image N of $\partial V \times S^1$ under the identification map, looks like an open book (Figure 1).

The fibers of V define the 'pages' and N, a closed codimension 2 submanifold is called the 'binding'. Every point $x \notin N$ lies on one and only one page and the boundary of each page coincides with N.

DEFINITION. A closed manifold is an open book if it is diffeomorphic to one of those just obtained.

Hence an open book is represented by a page V and a self-diffeomorphism $h: V \to V$, which restricts to the identity on ∂V. If $h_*: H_*(V, Z) \to H_*(V, Z)$ is the identity, we say that the open book decomposition has no monodromy.

AMS (MOS) subject classifications (1970). Primary 57A15, 57D15; Secondary 57C20, 57D65, 57D80.

Key words and phrases. Open book decomposition, minimal handle decomposition, h-cobordism, pages, binding, monodromy.

[1] Supported in part by National Science Foundation grant GP-7952X3.

[2] Also, J. P. Alexander (Ph.D. Thesis, University of California, Berkeley, 1971).

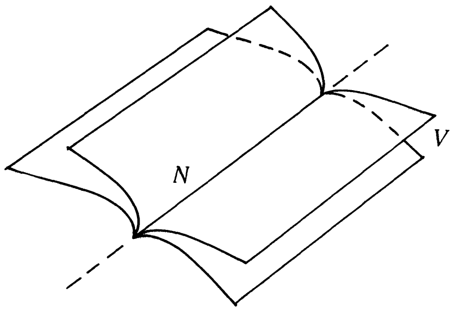

FIGURE 1

EXAMPLES. (a) Fibered knots. These are open book decompositions of spheres, where the bindings are also spheres.

(b) Milnor's Fibration Theorem [**4**] gives many nontrivial open book decompositions of S^{2k+1}, whose monodromy is interesting.

(c) Let V be any compact manifold with $\partial V \neq \emptyset$; in $V \times I$ identify each interval (x, t), $x \in \partial V$, $t \in I$, to a point $(x, \frac{1}{2})$ obtaining a manifold with boundary, W; let $N \subset \partial W$ be the image of $\partial V \times I$ under the identification, which divides ∂W into $(\partial W)_+$ and $(\partial W)_-$; if $h:(\partial W, (\partial W)_+, (\partial W)_-) \to (\partial W, (\partial W)_+, (\partial W)_-)$ is a diffeomorphism of triples, then $W \cup_h W$ has an open book decomposition with binding N.

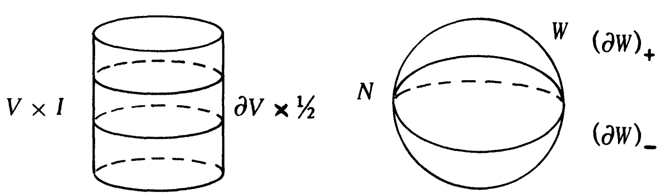

FIGURE 2

In 1923, Alexander [**1**] proved: Every orientable 3-manifold is an open book. We extend Alexander's theorem to dimensions >6.

OPEN BOOK THEOREM. *Let $n > 6$ and M be a closed, simply-connected n-manifold.*

Part 1. (a) *If $n \not\equiv 0 \bmod 4$, M is an open book.*

(b) *If $n \equiv 0 \bmod 4$, M is an open book if and only if the index $\tau(M) = 0$.*

Furthermore, our pages and bindings will also be simply-connected and $H_i(V, Z) = 0$ for $i > [n/2]$. We now normalize the open book definition by requiring that $H_i(V, Z) = 0$ for $i > [n/2]$.

Part 2. (a) *For odd* n, M *has an open book decomposition with no monodromy if and only if* $H_k(M, Z)$ *has no torsion* ($n = 2k + 1$).

(b) *If* $n \equiv 2 \bmod 4$, M *always has an open book decomposition with no monodromy.*

(c) *If* $n \equiv 0 \bmod 4$ *and* M *is of type* I, M *has an open book decomposition with no monodromy if and only if* $\tau(M) = 0$.

(d) *If* $n \equiv 0 \bmod 4$ *and* M *is of type* II, M *has an open book decomposition with no monodromy, if and only if* $\tau(M) = 0$ *and the middle-dimensional Wu class* $V_{n/2}(M) = 0$.

REMARK. Recall that M is called of type II if all numbers in the diagonal of its intersection matrix, with respect to some basis of $H_m(M, Z)$ mod torsion, are even; of type I, if it is not of type II ($n = 2m$).

2. **Proof of part I for** $n = 2k + 1 > 5$. The proof of this case already illustrates all the ideas involved in proving our theorem. Following Smale [**5**, Theorem 8.1], fix a minimal handle decomposition of M^{2k+1} and let W_1 be constructed by all i-handles for $i \leqq k$ and W_2 with all such i-handles of the dual handle decomposition. Then $M = W_1 \cup W_2$, $\partial W_1 = \partial W_2 = W_1 \cap W_2 = E$ (see Figure 3) and there exist k-dimensional subcomplexes $K_l \subset W_l$ ($l = 1, 2$) such that these inclusions are homotopy equivalences.

ASSERTION. There exists a k-complex $K \subset E \subset \partial W_1 = \partial W_2$ such that *both* inclusions $K \subset W_l$ are homotopy equivalences.

PROOF. Let $i_l : E \to K_l$ be the maps defined by $E \subset W_l \cong K_l$. Suppose we found a map $c : K_1 \to E$ such that both $i_1 c : K_1 \to K_1$ and $i_2 c : K_1 \to K_2$ are homotopy equivalences. Then, since $\dim E = 2k$, if we put c in general position, the only singularities will be transverse self-intersections of k-simplices of K_1. By a well known method, due to Stallings (embedding 'up to homotopy', see [**6**] for example), we can attach 2-disks to the image of c to obtain a complex $K \subset E = \partial W_1$, which is homotopically equivalent to K_1; here the condition $n = 2k + 1 > 5$ is used.

In order to find c notice the following (all homology groups are taken over the integers and $l = 1, 2$):

(a) $H_k(K_l)$ are free and, since the W_l are defined with respect to a minimal handlebody decomposition, by duality, they have the same number of generators.

(b) By duality, $H_i(W, E) = 0$ for $i \leqq k$ and, by the relative Hurewicz theorem, $\Pi_i(W_l, E) = 0$ for $i \leqq k$.

(c) By (b), the $i_{l*} : H_i(E) \to H_i(K_l)$ are isomorphisms for $i < k$ and epimorphisms for $i = k$.

(d) From (c) and the relative Hurewicz theorem, it follows that the i_l are $(k - 1)$-connected maps; i.e., if we consider them to be fiber maps (up to homotopy) with fibers F_l, then $\Pi_i(F_l) = 0$ for $i < k$.

By (b), there exists a map $c': K_1 \to E$ such that the diagram

$$\begin{tikzcd} W_1 \arrow[r, phantom, "\supset"] & E \\ K_1 \arrow[u, phantom, "\wr\|"] \arrow[ur, "c'"'] & \end{tikzcd}$$

commutes up to homotopy and so $i_1c': K_1 \to K_1$ induces isomorphisms in $H_*(K_1)$ and is therefore a homotopy equivalence; by composing with a homotopy equivalence we can suppose that i_1c' is homotopic to the identity $K_1 \to K_1$ and so there exists at least one cross section of the fibering $F_1 \to E \to K_1$. We wish to change this cross section, leaving it fixed on the $(k-1)$-skeleton of K_1, to a cross section $c: K_1 \to E$ such that $i_2c: K_1 \to K_2$ is also a homotopy equivalence. But the difference cocycle of two such cross sections lies in $H^k(K_1, \Pi_k(F))$, where K_1 is k-dimensional and $\Pi_i(F) = 0$ for $i < k$ (by (d)). Hence we are faced with a primary obstruction problem: If we can find any homomorphism $c_*: H_k(K_1) \to H_k(K_2)$ such that $i_{1*}c_* =$ identity and $i_{2*}c_*$ is an isomorphism, then we can change our cross section c', without changing it on the $(k-1)$-skeleton of K_1, to a cross section $c: K_1 \to E$ which induces c_*. By Whitehead's theorem, i_2c will then be a homotopy equivalence. I thank W. D. Neumann for the proof of the following algebraic lemma. Let F_n be the free abelian group of rank n and G a finitely generated abelian group; suppose i_1 and i_2 are epimorphisms $G \to F_n$;

(1)
$$\begin{tikzcd} & G \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ F_n \arrow[ur, dashed, "c"'] \arrow[rr, dashed, "i_2c"] & & F_n \end{tikzcd}$$

consider the problem of finding a homomorphism $c: F_n \to G$ such that $i_1c =$ identity and i_2c is an isomorphism.

Lemma. *c always exists, if we are allowed to stabilize, i.e. if we consider the diagram*

$$\begin{tikzcd} & G + F_n + F_n \arrow[dl, "i_1' = i_1 + p_1"'] \arrow[dr, "i_2 + p_2 = i_2'"] & \\ F_n + F_n & & F_n + F_n \end{tikzcd}$$

instead of (1). *Here p_1 and p_2 denote the projections $F_n + F_n \to F_n$.*

Proof. Since F_n is free, there exist homomorphisms $g_1, g_2: F_n \to G$ such that $i_1g_1 = \text{identity} = i_2g_2$; define $c: F_n + F_n \to G + F_n + F_n$ by $c(x, y) = (g_1x + g_2y - g_1i_1g_2y, y, x - i_1g_2y)$, then $i_2'c(x, y) = (x, y)$ and $i_2'c: F_n + F_n \to F_n + F_n$ is an isomorphism, because it is an epimorphism:

$$i_2'c(y - i_1g_2i_2g_1y + i_1g_2x, x - i_2g_1y) = (x, y).$$

By (a) and (c) our diagram

$$\begin{tikzcd} & H_k(E) \arrow[dl, "i_{1*}"'] \arrow[dr, "i_{2*}"] & \\ H_k(K_1) & & H_k(K_2) \end{tikzcd}$$

satisfies the hypothesis of the lemma; furthermore, we can suppose it has been stabilized by taking connected sums (along E and $S^k \times S^k$) of $M^{2k+1} = W_1 \cup W_2$ with $S^{2k+1} = S^k \times D^{k+1} \cup D^{k+1} \times S^k$ a certain number of times. We can therefore construct c_* above and follow it up geometrically to obtain a map $c: K_1 \to E$, such that both i_1c and i_2c are homotopy equivalences, which proves our assertion.

Let V be a regular neighborhood of K in E. By hypothesis, K has codimension 3 and so ∂V is also simply connected.

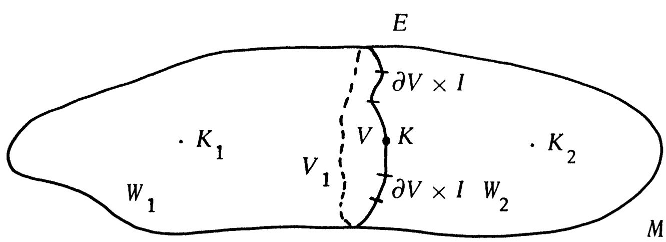

FIGURE 3

Take a collar neighborhood $\partial V \times I$ of V in E and denote by V_1 the closure of the complement of $V \cup (\partial V \times I)$ in E and regard W_1 and W_2 as relative cobordisms between V and V_1 (see Figure 3). $\partial V \times I$ is a product cobordism between ∂V and ∂V_1 and the *Assertion* implies that both W_1 and W_2 are relative h-cobordisms.

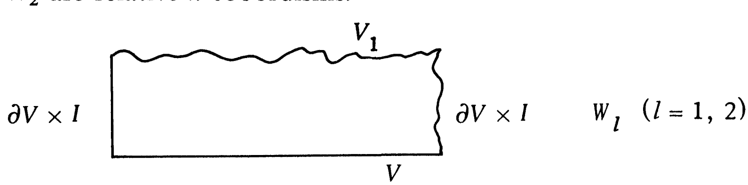

FIGURE 4

Hence, by the relative h-cobordism theorem $W_1 = V \times I = W_2$ and Figure 3 changes into

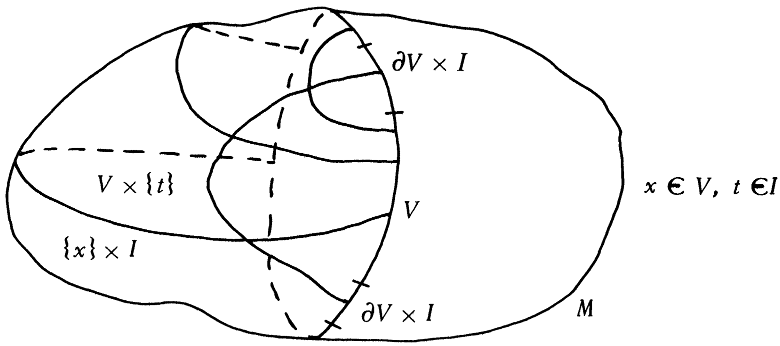

FIGURE 5

which, if we make our collar neighborhood $\partial V \times I$ smaller and smaller, changes into

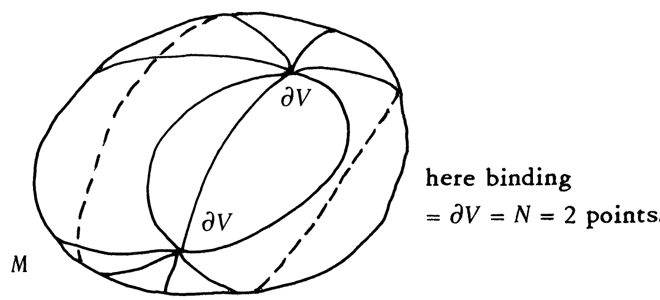

FIGURE 6

In other words, M can be obtained as in Example (c) of §1. Hence M is an open book with binding diffeomorphic to ∂V, with simply connected pages V such that $H_i(V, Z) = 0$ for $i > [n/2]$.

REMARK 1. For the proof of the case $n = 2k > 6$ of Part 1 we split M as in Chapter II of [**8**]: $M = W_1 \cup W_2$, where $H_i(W_l) = 0$ for $i > k$ $(l = 1, 2)$ and the intersection forms of W are $\equiv 0$ with respect to any coefficient group; the techniques of [**7**] then show that $W = V \times I$. Concerning Part 2 see [**9**] and Chapter II of [**8**] and for the complete details see [**10**]. Notice that all the above also hold in the piecewise-linear category.

REMARK 2. Recently I. Tamura independently found open book decompositions for a large subclass of simply-connected manifolds of $\tau = 0$ and dimension >6 and N. A'Campo (to appear in Comment. Math. Helv.) proved that every simply-connected 5-manifold has an open book decomposition with binding S^3.

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