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A HOMOTOPY CLASSIFICATION OF 2-COMPLEXES WITH FINITE CYCLIC FUNDAMENTAL GROUP

BY MICHEAL N. DYER AND ALLAN J. SIERADSKI

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For an arbitrary positive integer n, let Z_n denote the cyclic group of order n, and let $P_n = S^1 \cup_n e^2$ be the pseudo-projective plane of order n.

THEOREM. *Let X be a connected finite 2-dimensional CW-complex with fundamental group Z_n. Then*

(1) *X has the homotopy type of the sum $P_n \vee S^2 \vee \cdots \vee S^2$ of the pseudo-projective plane P_n and rank $H_2(X)$-copies of the 2-sphere S^2.*

(2) *There is a homotopy equivalence $f: X \to P_n \vee S^2 \vee \cdots \vee S^2$ realizing any prescribed Whitehead torsion $\tau(f) \in \mathrm{Wh}(Z_n)$.*

The result (1) was established in the prime order case by W. H. Cockcroft and R. G. Swan [**3**]. The work of P. Olum on the self-equivalences of the pseudo-projective plane P_n ([**6**], [**7**]) shows that every element of the Whitehead group Wh (Z_n) is realized as the torsion of some self-equivalence $P_n \to P_n$, so that (2) is a consequence of (1).

COROLLARY. *For connected finite 2-dimensional CW-complexes with finite cyclic fundamental group, homotopy type and simple homotopy type coincide.*

This generalizes to the nonprime order case a recent observation of W. H. Cockcroft and R. M. F. Moss [**2**].

SKETCH OF A PROOF OF THE THEOREM. Each CW-complex under consideration has the simple homotopy type of a complex P that is modeled in an obvious fashion on some presentation $\mathscr{P} = \langle a_1, \ldots, a_k : r_1, \ldots, r_m\rangle$ ($m \geqq k$) of the cyclic group Z_n. There are Nielsen transformations which reduce such a presentation to one of pre-Abelian form [**5**, p. 140]

$$\mathscr{Q} = \langle b_1, \ldots, b_k : b_1 W_1, \ldots, b_{k-1} W_{k-1}, b_k^n W_k, W_{k+1}, \ldots, W_m\rangle,$$

where the exponent sum of each word W_i with respect to each generator b_j is zero. Moreover, this Nielsen reduction $\mathscr{P} \to \mathscr{Q}$ corresponds to a simple homotopy equivalence $P \to Q$ of the associated topological models. Associated with each topological model P of a presentation $\mathscr{P}$ is the cellular chain complex $C_*(\tilde{P})$ of its universal covering $\tilde{P}$; the chain groups are free Z_n-modules which we give preferred bases according to a

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specific natural system. The chain complex $C_* = C_*(\tilde{Q})$ with its preferred bases is

$$\begin{array}{ccccc} C_2(\tilde{Q}) & & C_1(\tilde{Q}) & \partial_1 & C_0(\tilde{Q}) \\ \| & & \| & \| & \| \\ \{u_1, \ldots, u_m\} & \xrightarrow{\partial_2} & \{v_1, \ldots, v_k\} & \xrightarrow{(0,\ldots,0,x-1)} & \{z\} \end{array}$$

where $\{\ ,\ldots,\ \}$ is the free Z_n-module with the enclosed basis, and x is the generator of the multiplicative cyclic group Z_n.

Using Jacobinski's cancellation theorem for projective Z_n-modules ([4], [8, p. 215], [9, p. 178]), it is possible to choose a new basis $w_1, \ldots, w_m$ for the chain group $C_2 = C_2(\tilde{Q})$ such that the matrix of the boundary operation $\partial_2 : C_2(\tilde{Q}) \to C_1(\tilde{Q})$ with respect to this new basis for C_2 and the old basis $v_1, \ldots, v_k$ for C_1 is

$$A = \begin{pmatrix} 1 & & & 0 & 0 & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ 0 & & & 1 & 0 & & \\ 0 & & & 0 & N & 0 & 0 \end{pmatrix}$$

where the identity block is a $(k-1) \times (k-1)$ matrix and where $N = 1 + x + \cdots + x^{n-1}$ is in the integral group ring of Z_n. The chain complex C_* with the new preferred basis for C_2 takes the form

$$\begin{array}{ccccc} C_2 & & C_1 & & C_0 \\ \| & & \| & & \| \\ \{w_1, \ldots, w_m\} & \xrightarrow{A} & \{v_1, \ldots, v_k\} & \xrightarrow{(0,\ldots,x-1)} & \{z\}. \end{array}$$

With these preferred bases, the chain complex C_* is realizable as the cellular chain complex $C_*(\tilde{R})$ of the universal covering $\tilde{R}$ of the complex R modeled on the presentation $\mathcal{R} = \langle c_1, \ldots, c_k : c_1, \ldots, c_{k-1}, c_k^n, 1, \ldots, 1\rangle$ with $m - k$ trivial relators. The identity map between the chain complexes $C_*(\tilde{R})$ and $C_*(\tilde{Q})$ can be realized by a map $f: R \to Q$ that is necessarily a homotopy equivalence. This completes the proof of the theorem since the space R modeled on the presentation $\mathcal{R}$ has the simple homotopy type of the sum $P_n \vee S^2 \vee \cdots \vee S^2$ of the pseudo-projective plane P_n and $m - k$ copies of the 2-sphere S^2.

Full details of these and related results will appear elsewhere.

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Department of Mathematics, University of Oregon, Eugene, Oregon 97403