FUNCTION THEORETIC METHODS FOR HIGHER ORDER,
ELLIPITC EQUATIONS IN THREE
INDEPENDENT VARIABLES

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Introduction. S. Bergman [1] and I. N. Vekua [5] have developed
function theoretic methods for treating analytic, elliptic equations in two
variables. In particular, they have developed integral operator methods
for the normalized, second order equation

\[ \Delta u + a u_x + b u_y + c u = 0, \]

and the fourth order equation

\[ \Delta \Delta u + a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = 0. \]

Colton [2], [3] has shown recently how one may extend the results of
Bergman and Vekua for the second order case when the coefficients and
solution are to be dependent on three variables. Colton’s method is based
on some earlier work of Tjong [4].

In this note we wish to announce that one may extend the ideas used by
Colton and Tjong to treat also equations of higher order, which depend
on three independent variables. We remark that this is the first time a
function theoretic method has been developed for a fairly general, higher
order equation in three independent variables. To simplify our presenta­
tion, and because of lack of space we will announce our results merely for
the case

\[ \Delta \Delta u + Q(x, y, z) u = 0; \]

the more general case, corresponding to equations (2) and the higher
order analogues may be treated in the same manner.

The generating kernels. The approach used by Bergman and Vekua has
been to continue the elliptic equation into the complex domain where it
is formally hyperbolic. By introducing the variables \( X = x, Z = \frac{1}{2}[y + iz], \)


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\[ Z^* = \frac{1}{2}[-y + iz] \]
equation (3) may be written in the form

\[ (4) \quad \frac{\partial^4 U}{\partial Z^2 \partial Z^{*2}} - 2 \frac{\partial^4 U}{\partial X^2 \partial Z \partial Z^{*}} + \frac{\partial^4 U}{\partial X^4} + Q(X, Z, Z^*) U = 0. \]

It is convenient for our exposition to introduce the variables \([4], [2]\]

\[ \xi_1 = 2\zeta Z, \quad \xi_2 = X + 2\zeta Z, \quad \xi_3 = X + 2\zeta^{-1} Z^*, \]

where \(\zeta\) is a complex variable such that \(1 - \varepsilon < |\zeta| < 1 + \varepsilon, 0 < \varepsilon < \frac{1}{2}\).

Furthermore, let \(E^*(\xi_1, \xi_2, \xi_3, t) \equiv E(X, Z, Z^*, \zeta, t)\), and \(E_i^* \equiv \partial E^*/\partial \xi_i\) \((i = 1, 2, 3)\), \(E_t^* \equiv \partial E^*/\partial t\). Suppose \(E^*\) is a solution of the partial differential equation

\[ (5) \quad \frac{\partial^2 E^*}{\partial t^2} + \frac{\partial^2 E^*}{\partial \zeta^2} + 6 \frac{\partial^2 E^*}{\partial \zeta \partial \zeta^*} - 8 E_{12}^* - 4 E_{123}^* \]

\[ - 8 E_{133}^* - 4 E_{233}^* + 16 E_{113}^* + 16 E_{123}^* \]

\[ - \frac{2(1 - t^2)}{\mu t} (E_{122t}^* + E_{133t}^* - 4 E_{113t}^* - 2 E_{123t}^*) \]

\[ + \frac{2}{\mu t^2} (E_{122}^* + E_{133}^* - 4 E_{113}^* - 2 E_{123}^*) \]

\[ + \frac{1}{\mu^2 t^2} (1 - t^2)^2 E_{11t}^* + \frac{E_{11t}^*}{t} (-3 + 3t^4) + \frac{3 E_{11}^*}{t^2} \]

\[ + Q^* E^* = 0, \]

for \((x, y, z) \in \mathcal{G} \subset \mathbb{C}^3\), where \(\mathcal{G}\) is some neighborhood of the origin, \(|t| \leq 1\), and \(\zeta\) is in the annulus described above. Then

\[ (6) \quad U(X, Z, Z^*) = \frac{1}{2\pi i} \int_{|t| = 1} \int_{\gamma} E(X, Z, Z^*, \zeta, t) f(\omega, \zeta) \frac{dt}{(1 - t^2)^{1/2}} \frac{d\zeta}{\zeta}, \]

where \(\gamma\) is a path joining \(t = -1\) and \(t = +1\), is a (complex valued) solution of equation (4) which is regular in a neighborhood of the origin in \(X, Z, Z^*\) space.

One may prove that solutions of (5) exist using the method of majorants. Indeed, we may find two solutions of the form

\[ E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \]

\[ \hat{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \frac{\xi_1}{2\zeta} + \sum_{n=1}^{\infty} t^{2n} \mu^n q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \]

where the coefficients \(p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)\) and \(q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)\) are uniquely determined by the following:
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\( p^{(1)}(\xi_1, \xi_2, \xi_3, \zeta) = 0 = q^{(1)}(\xi_1, \xi_2, \xi_3, \zeta), \)

\( p^{(2)}_{11}(\xi_1, \xi_2, \xi_3, \zeta) = -\frac{1}{3}Q^*(\xi_1, \xi_2, \xi_3, \zeta), \)

\( q^{(2)}_{11}(\xi_1, \xi_2, \xi_3, \zeta) = -\frac{1}{3}(\xi_1/2\zeta)Q^*(\xi_1, \xi_2, \xi_3, \zeta), \)

and for \( n \geq 1 \) both \( p^{(n)} \) and \( q^{(n)} \) satisfy

\[
p^{(n+2)}_{11} = \frac{1}{(2n+1)(2n+3)} \left[ 2(2n+1)(p^{(n+1)}_{1221} + p^{(n+1)}_{133} - 4p^{(n+1)}_{133} - 2p^{(n+1)}_{123}) - \right. \\
\left. p^{(n)}_{2222} - p^{(n)}_{3333} - 6p^{(n)}_{2233} + 4p^{(n)}_{2233} + 4p^{(n)}_{2333} + 8p^{(n)}_{1223} + 8p^{(n)}_{1333} - 16p^{(n)}_{1233} - 16p^{(n)}_{1133} - Q^*p^{(n)} \right],
\]

such that, for \( n = 1, 2, \ldots, \)

\[
p^{(n)}(0, \xi_2, \xi_3, \zeta) = 0 = q^{(n)}(0, \xi_2, \xi_3, \zeta),
\]

\[
p^{(n)}_1(0, \xi_2, \xi_3, \zeta) = 0 = q^{(n)}_1(0, \xi_2, \xi_3, \zeta).
\]

**Complete families of solutions.** The particular choice of the generating kernels indicated by (7) are of importance in showing that representations of the form

\[
u(x, y, z) = \Re P^2_3 \{ f, \hat{f} \},
\]

\[
P^2_3 \{ f, \hat{f} \} \equiv \frac{1}{2\pi i} \int_{|\zeta| = 1} \int_{\gamma} \left[ E(X, Z, Z^*, \zeta, t) f(\omega, \zeta) \frac{dt}{(1-t^2)^{1/2}} \frac{d\zeta}{\zeta} \right. \\
\left. + \hat{E}(X, Z, Z^*, \zeta, t) \hat{f}(\omega, \zeta) \frac{dt}{(1-t^2)^{1/2}} \frac{d\zeta}{\zeta} \right],
\]

are maps of pairs of holomorphic functions onto \( \mathcal{C}^4 \), real solutions of (3).

Indeed, we are able to prove the

**Theorem.** Let \( u(x, y, z) \) be a real valued \( \mathcal{C}^4 \) solution of equation (3) in some neighborhood of the origin in \( \mathbb{R}^3 \). Then there exists a pair of analytic functions of two complex variables \( \{ f(\mu, \zeta), \hat{f}(\mu, \zeta) \} \) which are regular for \( \mu \) in some neighborhood of the origin and \( |\zeta| < 1 + \varepsilon, \varepsilon > 0 \), such that locally \( u(x, y, z) = \Re P^2_3 \{ f, \hat{f} \} \).

In particular, denote by \( U(X, Z, Z^*) \) the extension of \( u(x, y, z) \) to the \( X, Z, Z^* \) space and let

\[
f(\mu, \zeta) = -\frac{1}{2\pi} \int_{\gamma} g(\mu(1 - t^2), \zeta) dt/t^2,
\]

\[
\hat{f}(\mu, \zeta) = -\frac{1}{2\pi} \int_{\gamma} \hat{g}(\mu(1 - t^2), \zeta) dt/t^2,
\]

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where \( y' \) is a rectifiable arc joining the points \( t = -1 \) and \( t = +1 \) and not passing through the origin. Here

\[
(14) \quad g(\mu, \zeta) = 2 \frac{\partial}{\partial \mu} \left[ \mu \int_{0}^{1} U(t\mu, 0, (1 - t)\mu\zeta) \, dt \right] - U(\mu, 0, 0),
\]

\[
(15) \quad \hat{g}(\mu, \zeta) = 2 \frac{\partial}{\partial \mu} \left[ \mu \int_{0}^{1} U_2(t\mu, 0, (1 - t)\mu\zeta) \, dt \right] - \zeta \frac{\partial}{\partial \mu} \left[ g(\mu/2, \zeta \mu) \right]
\]

\[
= g(\mu, 0) - \zeta \int_{0}^{\mu} U(\mu, 0, 0) \, d\mu.
\]

The previous theorem implies that we may generate a family of solutions with the Runge approximation property.

**Theorem III.** Let \( \mathcal{G} \) be a bounded simply connected domain in \( \mathbb{R}^3 \), and define

\[
U_{4n,m} = \text{Re} \, P^{(2)}_3 \{ \mu^{n\zeta m}, 0 \}, \quad 0 \leq n < \infty, m = 0, \ldots, n,
\]

\[
U_{4n+1,m} = \text{Re} \, P^{(2)}_3 \{ 0, \mu^{n\zeta m} \}, \quad 0 \leq n < \infty, m = 0, \ldots, n,
\]

\[
U_{4n+2,m} = \text{Im} \, P^{(2)}_3 \{ \mu^{n\zeta m}, 0 \}, \quad 0 \leq n < \infty, m = 0, \ldots, n,
\]

\[
U_{4n+3,m} = \text{Im} \, P^{(2)}_3 \{ 0, \mu^{n\zeta m} \}, \quad 0 \leq n < \infty, m = 0, \ldots, n,
\]

where "Im" denotes "take the imaginary part". Then the set \( \{ U_{n,m} \} \) is a complete family of solutions for equation (3) in the space of real valued \( \mathcal{C}^4 \) solutions of (3) defined in \( \mathcal{G} \).

**References**


