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THE COHOMOLOGY OF RESTRICTIONS OF THE $\bar{\partial}$ COMPLEX

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The local problem for complex vector fields (see for example Kohn [**3**]) can be summarized as follows: We are given a family $L_1, \ldots, L_m$ of vector fields on some neighborhood U of the origin in R^n:

$$L_i = \sum a_{ij} \frac{\partial}{\partial x_j}; \quad i = 1, \ldots, m,$$

where the a_{ij} are complex valued C^∞ functions on U. We assume that this family is closed under Lie brackets, i.e.

$$[L_i, L_j] = \sum d_{ij}^k L_k; \qquad i, j = 1, \ldots, m.$$

We then look at the equations

$$L_i(u) = f_i; \qquad i = 1, \ldots, m, \tag{1}$$

where the f_i are C^∞ functions on U, and try to give conditions on the L_i and the f_i so that a solution u should exist on maybe a smaller neighborhood of 0. We might also ask about the regularity properties of the solution u.

If we further assume that the L_i are linearly independent at each point of U, we can consider them as a basis for the sections of a vector bundle $\mathscr{L}$ on U, and we obtain a complex

$$C_0^\infty(V) \overset{D_0}{\to} \Gamma(\mathscr{L}^*, V) \overset{D_1}{\to} \Gamma(\mathscr{L}^* \wedge \mathscr{L}^*, V) \overset{D_2}{\to} \ldots, \qquad V \subset U.$$

Now equation (1) becomes $D_0(u) = f$ (see [**3**]).

For example, if M is a C^∞ submanifold of R^n, and d is the exterior derivative operator, the solution of the local problem is given by the Poincaré lemma and similarly, if M is a complex manifold, the solutions for the operators ∂ or $\bar{\partial}$ is given by the Dolbeault-Grothendieck lemma. In these two cases if M is compact we know that the cohomology spaces $H^i(M)$ $(i > 0)$ are finite dimensional. A more difficult example is obtained if we take the restriction of the $\bar{\partial}$ (or the ∂) operator to a C^∞ real submanifold of a complex manifold (see [**2**], [**3**], [**4**] and [**6**]).

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In general, a great deal can be said ([2], [3] and [7]) if we make the additional assumption:

(A) If in addition to the L_i we also consider the vector fields $\bar{L}_i$ and $[L_i, \bar{L}_j]$ $(i, j = 1, \ldots, m)$, all these together span a vector bundle $\mathscr{M}$, and the codimension of $\mathscr{L} \oplus \overline{\mathscr{L}}$ in $\mathscr{M}$ is at most one.

Condition (A) is satisfied when we consider the restriction of the $\bar{\partial}$ operator to a C^∞ real hypersurface M of X^n $(n > 1)$, in which case $\mathscr{L}$ is the bundle whose sections are the complex tangent vector fields on M of type (0, 1). The analysis of this case (which as usual provides the clue for operators other than the $\bar{\partial}$) was given by Kohn in [1]. Let $D^{p,q}(M) = \Lambda^p \mathscr{L}^* \wedge \Lambda^q \overline{\mathscr{L}}^*$, and let $\bar{\partial}_M$ stand for the restriction of the $\bar{\partial}$ operator to M, and $\bar{\partial}_M^*$ for its Hilbert space adjoint.

THEOREM (KOHN). *If M is a C^∞ real compact hypersurface of X^n (i.e. $\operatorname{codim}_R M = 1$), and the Levi form of M has at least $\max(n - q, q + 1)$ eigenvalues of the same sign $\neq 0$ at each point of M, then there is a constant $C > 0$ such that*

$$\|\phi\|_{1/2} \leqq C(\|\bar{\partial}_M \phi\| + \|\bar{\partial}_M^* \phi\| + \|\phi\|) \quad \text{for all } \phi \in D^{p,q}(M). \tag{2}$$

Here $\|\cdot\|_{1/2}$ stands for the Sobolev $\frac{1}{2}$-norm. The estimate (2) implies that we have a harmonic theory, and in particular the cohomology space of type (p, q) is finite dimensional.

Now let M be a C^∞ real submanifold of a complex manifold X^n, and assume that $\mathscr{L}$, the complex vector fields on M of type (0, 1) is a bundle. Such submanifolds are called C-R submanifolds. If the real codimension of M in X is bigger than one, condition (A) will not in general be satisfied. Although we are not yet able to prove subelliptic estimates such as (2) in this case, we can prove a Poincaré lemma for the induced operator $\bar{\partial}_M$, and finiteness theorems for the cohomology of compact C-R submanifolds.

THEOREM 1. *Let M be a C-R submanifold of a complex manifold X^n, and let $m = \dim_C \mathscr{L}(M)$ (in the case of a real hypersurface we have $m = n - 1$). Suppose that $P \in M$ and that there is a C^∞ real hypersurface $S \subset X$ which contains M locally at P such that its Levi form restricted to $\mathscr{L}(M)_P$ has either at least $\max(m - q + 1, q + 1)$ eigenvalues of the same sign $\neq 0$, or at least $2q$ $(q > 0)$ pairs of eigenvalues of opposite sign.[2] Then there is a fundamental system of neighborhoods $\{V\}$ of P in M such that if $f \in D^{p,q}(V)$ and $\bar{\partial}_M(f) = 0$, there is $u \in D^{p,q-1}(V)$ with $\bar{\partial}_M(u) = f$.*

[2] The condition "$2q$ pairs of eigenvalues of opposite sign" in the case when M is a hypersurface does not appear explicitly in Kohn's original paper [1], but can be obtained without difficulty with his proof.

THEOREM 2. *If M is a compact C-R submanifold and the above condition is satisfied at each point of M, the cohomology space of type (p, q) is finite dimensional.*

The proof of the above theorems is based on the techniques developed in [**4**] (see also [**5**]). We construct suitable tubular local neighborhoods T_ε of M ($\varepsilon > 0$) with the property that given a $\bar{\partial}$-closed form f on T_ε, one can find another $\bar{\partial}$-closed form g in T_ε such that $g = f$ on $T_{\varepsilon/2}$ and $\|g\| \leqq C(\varepsilon)\|f\|$ where $C(\varepsilon)$ is an $O(\varepsilon^{-k})$ for some fixed k as $\varepsilon \to 0$, and such that the equation $\bar{\partial}u = g$ can be solved on T_ε (this is the case if for example g has compact support in T_ε). The details of the proof will appear elsewhere.

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