BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

ON MINIMAL IMMERSIONS OF S^2 IN S^{2m}

BY JOAO LUCAS MARQUES BARBOSA[1]

Communicated by S. S. Chern, August 9, 1972

1. **Introduction.** Let $x:S^2 \to S^{2m}(1)$ be a minimal immersion of the 2-sphere into the unit sphere of dimension $2m$. Following S. S. Chern [**4**] we associate to x a certain holomorphic curve from S^2 with values in CP^{2m} (the complex projective space of dimension $2m$) called the directrix curve of the minimal immersion. (Note that in the induced metric, S^2 acquires a conformal structure.) This curve is rational, and the unique condition it must satisfy is that of being totally isotropic, i.e. if ξ is any of its local representations in homogeneous coordinates, then ξ satisfies

$$(\xi, \xi) = (\xi', \xi') = \cdots = (\xi^{m-1}, \xi^{m-1}) = 0$$

where $(,)$ denotes the symmetric product in C^{2m+1}.

Chern proved, for the case $m = 2$, that the simple condition of total isotropy completely characterizes the set of directrix curves among all holomorphic ones from S^2 into CP^{2m} if, instead of minimal immersions, we consider generalized minimal immersions. In the paper for which this is an announcement, we generalize this result and obtain further geometrical information about the corresponding minimal immersion. Complete proofs will appear elsewhere.

The first systematic study of this subject was made by E. Calabi who also associated, implicitly in (2) and explicitly in (3), a holomorphic curve η to the minimal immersion x. It turns out that η is the $(m - 1)$th associated curve of the directrix curve. This observation has allowed us to unify the approaches developed previously by Calabi and Chern.

2. **Definitions and preliminary remarks.** Let $x:S^2 \to S^{2m}$ be a differentiable map into the unit $2m$-sphere and let z be a local isothermal parameter in S^2 relative to the induced metric. Set $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. Then we denote by

$$ds^2 = 2F|dz|^2, \quad F = (\partial x, \bar{\partial} x), \quad \text{the metric of } S^2,$$
$$\omega = iF dz \wedge d\bar{z}, \quad \text{the area form, and by}$$
$$K = -\partial\bar{\partial} \log(F)/F, \quad \text{the Gauss curvature.}$$

The minimality of x is then equivalent to the equation

$$\partial\bar{\partial} x = -Fx.$$

AMS (MOS) subject classifications (1970). Primary 53B25, 32C25, 58D10; Secondary 14H45, 14H99, 14M15, 53C40.

[1] Research supported by Conselho Nacional de Pesquisas do Brazil and NSF Grant GP-34785X.

We will allow F to have isolated zeros, whereby the immersion is called generalized. We consider at each point of S^2 the complex subspace $V(x)$ of C^{2m+1} spanned by $\{\partial x, \partial^2 x, \partial^3 x, \ldots\}$. One can prove that $V(x)$ is totally isotropic and perpendicular to x. Furthermore, if we assume that x does not lie in any lower dimensional subspace of R^{2m+1}, then $V(x)$ can be represented locally, in Plücker coordinates, by $\partial x \wedge \partial^2 x \wedge \cdots \wedge \partial^m x$ except for a finite number of points where this product is zero.

3. **The directrix curve.** Define $G_0, G_1, G_2, \ldots$ by the following recurrence formulae:

$$G_0 = x,$$

$$G_k = \bar{\partial}^k x - \sum_{j=1}^{k-1} a_k^j G_j, \qquad k = 1, 2, 3, \ldots,$$

where the a_k^j are chosen in such a way that each G_k is perpendicular to all the previous ones. Then one can prove

LEMMA 1.

$$\bar{\partial} G_k = G_{k+1} + (\bar{\partial} \log|G_k|^2) G_k,$$

$$\partial G_k = -|G_k|^2 G_{k-1}/|G_{k-1}|^2, \quad \text{for } k \geqq 1.$$

Certainly $G_{m+k} = 0$ and so this lemma gives

$$\bar{\partial} G_m = (\bar{\partial} \log|G_m|^2) G_m.$$

Since $G_m \not\equiv 0$, we may then use G_m as a local definition for a holomorphic curve from S^2 into CP^{2m} that we call the directrix curve of x. This curve is totally isotropic and does not lie in any complex hyperplane of CP^{2m}. If ξ denotes the directrix curve, we shall denote by $\xi_1, \xi_2, \xi_3, \ldots$ its associated curves. For definitions see [**6**, p. 71].

It is quite natural to ask if we can reverse the process described above and, starting from an arbitrary totally isotropic holomorphic curve $\xi: S^2 \to CP^{2m}$ that is not contained in any complex hyperplane of CP^{2m}, somehow construct a minimal immersion $x: S^2 \to S^{2m}$ that has ξ as its directrix curve. This is indeed possible. The way of doing this is by considering, locally, a vector defined by

$$\psi = \xi \wedge \xi' \wedge \cdots \wedge \xi^{m-1} \wedge \bar{\xi} \wedge \bar{\xi}' \wedge \cdots \wedge \bar{\xi}^{m-1}$$

and then observing that ψ is either real or total imaginary. Set $\tilde{\psi}$ equal to ψ if ψ is real, or $-i\psi$ if ψ is totally imaginary. Then we prove the following proposition:

PROPOSITION 1. *$\tilde{\psi}/|\tilde{\psi}|$ is independent of the particular local coordinates used, and so it defines a map x from S^2 into S^{2m}. Furthermore we have,* *relative to a local coordinate z on S^2, that $(\partial x, \partial x) = 0$ and $\partial\bar{\partial}x$ is parallel to x.*

An immediate consequence of Proposition 1 is that the map $x = \tilde{\psi}/|\tilde{\psi}|$ is minimal. It is also possible to prove the following proposition which gives a criterion for the regularity of x.

PROPOSITION 2. *The map x, obtained by the above construction satisfies, in terms of the local coordinate z, the relation*

$$(\partial x, \bar{\partial} x) = |\xi_{m-1} \wedge \xi'_{m-1}|^2/|\xi_{m-1}|^4.$$

This implies that x and ξ_{m-1} are isometric and consequently x will be regular except for a finite number of points. Thus, x will be a generalized minimal immersion.

PROPOSITION 3. *Suppose $\xi, \zeta: S^2 \to CP^{2m}$ are totally isotropic curves which do not lie in a complex hyperplane of CP^{2m} and which give rise to the same minimal immersion $x: S^2 \to S^{2m}$ by the process described above. Then $\xi = \zeta$.*

Now Propositions 1, 2 and 3 can be put together to give a proof of the following theorem.

THEOREM 1. *There exists a canonical* 1-1 *correspondence between the set of generalized minimal immersions $x: S^2 \to S^{2m}$ which are not contained in any lower dimensional subspace of R^{2m+1} and the set of totally isotropic holomorphic curves $\xi: S^2 \to CP^{2m}$ which are not contained in any complex hyperplane of CP^{2m}. The correspondence is the one that associates to each minimal immersion x its directrix curve.*

The principal importance of this theorem is that it allows us to identify these two spaces; and the space of totally isotropic holomorphic curves is much easier to study.

4. **The area of a minimal immersion** $x: S^2 \to S^{2m}$. Let us look a bit more carefully at what Proposition 2 says. From the fact that x and ξ_{m-1} are isometric, we conclude that:

$$\text{Area}(x) = 2\pi \,\text{degree}(\xi_{m-1}).$$

This was already known to Calabi [**2**]. Since ξ_{m-1} is not a general curve in CP^{N-1}, $N = \binom{2m+1}{m}$, but a totally isotropic one, we may ask ourselves what values degree (ξ_{m-1}) can assume. In fact, if we represent by $\mathscr{H}_m$ the manifold whose points are m-dimensional totally isotropic subspaces of C^{2m+1}, we may think of ξ_{m-1} as a function with values in this space. It turns out that $\mathscr{H}_m$ is a Kähler submanifold of the corresponding projective space and also a homogeneous space given by $SO(2m+1)/U(m)$.

The following lemmas give some information about how $\mathscr{H}_m$ lies in the projective space.

LEMMA 2. *$H_2(\mathscr{H}_m, Z)$, the second homology group of $\mathscr{H}_m$ with integer coefficients, is isomorphic to Z.*

LEMMA 3. *We have, in a canonical way, $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \mathscr{H}_3 \subset \cdots \subset \mathscr{H}_m$. The space $\mathscr{H}_1$ is a generator for $H_2(\mathscr{H}_m, Z)$ and the degree of $\mathscr{H}_1$, as a submanifold of CP^{N-1}, is 2.*

Lemma 2 allows us to define a notion of degree for curves in $\mathscr{H}_m$ and by Lemma 3 we know that such a degree is half of the degree of the curve considered now in the projective space.

THEOREM 2. *The area of a generalized minimal immersion $x: S^2 \to S^{2m}$ must be a multiple of 4π.*

It is known that for x not totally geodesic, we have

$$\mathrm{Area}(x) \geqq 2\pi m(m + 1)$$

and we are able to exhibit examples for all allowed multiples of 4π. This makes this theorem the best possible.

5. **A rigidity theorem.** Certainly if we know one example of a totally isotropic holomorphic curve, we are able to construct immediately a whole family of them by considering its orbit under the action of the group $SO(2m + 1, C)$. We certainly would like to know how many of them are really different, that is, we would like to identify minimal immersions up to rigid motions of S^{2m} or even up to isometry. The latter sounds more interesting and one is naturally led to consider what relation there is between the directrix curves of two isometric minimal immersions $x, y: S^2 \to S^{2m}$.

PROPOSITION 4. *Let $x, y: S^2 \to S^{2m}$ be generalized minimal immersions which do not lie in any lower dimensional subspace of R^{2m+1}, and let $\xi, \zeta: S^2 \to CP^{2m}$ be the corresponding directrix curves. Then, x, y are isometric if and only if ξ, ζ are.*

The proof of this follows from the fact that x and ξ_{m-1} are isometric and from the following lemma which allows us to conclude that the metric of ξ is completely determined by the metric of x.

LEMMA 4. *Let $x: S^2 \to S^{2m}$ be a generalized minimal immersion which does not lie in any lower dimensional subspace of R^{2m+1}. Define $F_k = |\partial x \wedge \partial^2 x \wedge \cdots \wedge \partial^k x|^2$. Let $\xi: S^2 \to CP^{2m}$ be the directrix curve* *of x and let* Ω_k *be the curvature form of* ξ_k. *Then, if we define* $F_0 = 1$, *we have*

$$F_{k+1} = \frac{F_k}{F_{k-1}} (\partial\bar{\partial} \log(F_k) + F),$$

$$\Omega_k = \frac{i}{2\pi} (\partial\bar{\partial} \log(F_{m-k-1}) + F)\, dz \wedge d\bar{z}.$$

Now, by Calabi [**1**] there exists a unique unitary transformation U of C^{2m+1} such that $\zeta = U\xi$. Since U takes a totally isotropic curve into another one, U must be very special. In fact we have

LEMMA 5. *Let* $\xi, \zeta : C \to C^{2m+1}$ *be totally isotropic polynomials not lying in any lower dimensional complex subspace of* C^{2m+1}. *If there exists a linear transformation* U *of* C^{2m+1} *such that* $U\xi = \zeta$, *then* $U \in SO(2m+1, C)$.

From this we may conclude the following:

THEOREM 3. *Suppose* $x, y : S^2 \to S^{2m}$ *are minimal immersions which do not lie in any lower dimensional subspace of* R^{2m+1}. *Then x and y are isometric if and only if they differ by a rigid motion of* S^{2m}.

REFERENCES

1. E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23. MR **15**, 160.

2. ———, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geometry **1** (1967), 111–125. MR **38** #1616.

3. ———, *Quelques applications de l'analyse complexe aux surfaces d'aire minima*, Topics in Complex Manifolds, University of Montreal, Canada, 1968, pp. 59–81.

4. S. S. Chern, *On the minimal immersions of the two sphere in a space of constant curvature*, Problems in Analysis, Princeton, N.J., 1970, pp. 27–40.

5. ———, *On minimal spheres in the four sphere*, Studies and Essays (Presented to Yu-why Chen on his 60th Birthday, April 1, 1970), Math. Res. Center, Nat. Taiwan Univ., Taipei, 1970, pp. 137–150. MR **43** #3936.

6. Wu Hung-Hsi, *The equidistribution theory of holomorphic curves*, Ann. of Math. Studies, no. 64, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1970. MR **42** #7951.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Instituto de Matematica, Universidade Federal do Ceara, Ceara, Brazil.