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SOME VARIATIONAL PROBLEMS ON CERTAIN SOBOLEV SPACES AND PERFECT SPLINES

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The Sobolev space $W_\infty^{(n)}[a, b]$ is comprised of all functions f defined on $[a, b]$ where $f^{(n-1)}$ is absolutely continuous and the maximum norm of its nth derivative $\|f^{(n)}\|_\infty$ is finite. For ease of exposition, henceforth let $[a, b] = [0, 1]$. Let $x_1 < x_2 < \ldots < x_{n+r}$ be $n + r$ prescribed points in $[0, 1]$ and $\{\alpha_i\}_{i=1}^{n+r}$ given real numbers. Consider the subset $\mathscr{I}[\boldsymbol{x}, \boldsymbol{\alpha}]$ of $W_\infty^{(n)}$ consisting of all $f \in W_\infty^{(n)}$ interpolating the data $\{\alpha_i\}$ at $\{x_i\}$; i.e., $f \in W_\infty^{(n)}$ satisfies

$$f(x_i) = \alpha_i, \qquad i = 1, 2, \ldots, n + r. \tag{1}$$

A problem of interest in approximation theory is to characterize the function yielding

$$\min_{f \in \mathscr{I}[x, \alpha]} \|f^{(n)}\|_\infty \tag{2}$$

(for background on the problem see, e.g., Glaeser [**1**], Tihomirov [**5**], Schoenberg [**3**]). We will prove that the minimum is attained by a "perfect spline." A *perfect spline* of degree n with $r - 1$ knots in $[0, 1]$ is a spline polynomial of the special form

$$P(x) = c\left[x^n + 2\sum_{i=1}^{r-1} (-1)^i (x - \xi_i)_+^n\right] + \sum_{i=0}^{n-1} a_i x^i \tag{3}$$

where $c, a_0, \ldots, a_{n-1}$ are real constants and the knots $\{\xi_i\}$ obey the constraints $0 < \xi_1 < \xi_2 < \ldots < \xi_{r-1} < 1$.

Manifestly, the perfect spline $P(x)$ exhibits the property that its nth derivative, though changing sign at each knot ξ_i, maintains a constant absolute value, in this case $|c|n!$ The key to the solution of the problem (2) is the following interpolation theorem.

THEOREM 1. *Let $\{x_i\}_{i=1}^{n+r}$ be prescribed with $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{n+r} \leqq 1$ involving no coincident block exceeding n points. Let $\{\alpha_i\}_{i=1}^{n+r}$ be given real data. Then there exists a perfect spline $P(x)$ of the form* (3) *with at most $k - 1$ knots in* $[0, 1]$, *$k \leqq r$, such that*

$$P(x_i) = \alpha_i, \qquad i = 1, 2, \ldots, n + r. \tag{4}$$

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(When coincident x_i occurs then the extra interpolation conditions are interpreted as interpolation for appropriate successive derivatives at that x value.) The interpolating spline is not necessarily unique.

The proof of Theorem 1 appears to be rather deep. Our analyses use results equivalent to the theory of degree of mapping of nonlinear transformations, and further refined facts on the total positivity nature of the kernel $K(x, \xi) = (x - \xi)_+^n$. The very special case of Theorem 1 with $r = n$ and $x_1 = x_2 = \ldots = x_n = 0$, $x_{n+1} = x_{n+2} = \ldots = x_{2n} = 1$ can be obtained by exploiting a form of Taylor's expansion with remainder formula and some elegent functional analysis. This special case was treated by Glaeser [**1**].

An explicit determination of the interpolating perfect spline for the interpolation conditions $P(0) = P'(0) = \cdots = P^{(n-1)}(0) = 0$, $P(1) = 1$, $P'(1) = P''(1) = \cdots = P^{(n-1)}(1) = 0$ was given by Louboutin [**2**]. For this example,

$$P(x) = x^n + 2 \sum_{i=0}^{n-1} (-1)^i (x - \xi_i)_+^n$$

where the knots $\{\xi_i\}_{i=0}^{n-1}$ are identified as the zeros of $T_n(x)$ (the Tchebycheff polynomial of the first kind), specifically

$$\xi_\nu = \frac{1 - \cos(\pi\nu/n)}{2}, \qquad \nu = 1, 2, \ldots, n - 1.$$

A relatively easy consequence of Theorem 1 involving repeated reference to Rolle's theorem leads to the next assertion.

COROLLARY 1. *The perfect spline affirmed in Theorem* 1 *provides a minimum in* (2).

Further topological arguments involving mainly applications of the Brouwer fixed point theorem with heavy reliance on the result of Theorem 1 establishes the existence of the two special perfect splines described in the next theorem.

THEOREM 2. *Let $g(x)$ and $h(x)$ be continuous positive and nonpositive, respectively, on* $[0, 1]$. *Let n and r be given and suppose first that $n + r = 2m$. There exists two special perfect splines* $\tilde{P}(x)$ and $\underline{P}(x)$ *with r knots, each oscillating maximally between $g(x)$ and $h(x)$ in the following sense. $\tilde{P}(x)$ is characterized by the properties*

(i) $h(x) \leqq \tilde{P}(x) \leqq g(x)$ *on* $[0, 1]$;

(ii) *there exists* $2m + 1$ *points* $0 \leqq \tilde{z}_1 < \tilde{y}_1 < \tilde{z}_2 < \tilde{y}_2 < \tilde{z}_3 < \ldots < \tilde{z}_m < \tilde{y}_m < \tilde{z}_{m+1} \leqq 1$ *such that* $\tilde{P}(\tilde{y}_i) = h(\tilde{y}_i)$, $i = 1, 2, \ldots, m$ *and* $\tilde{P}(\tilde{z}) = g(\tilde{z}_i)$, $i = 1, 2, \ldots, m + 1$.

Similarly, the perfect spline $\underline{P}(x)$ *is characterized by the conditions* (i) *as above and* (ii') *asserting the existence of* $0 \leqq \underline{y}_1 < \underline{z}_1 < \underline{y}_2 < \underline{z}_2 < \cdots < \underline{z}_m < \underline{y}_{m+1} \leqq 1$ *where* $\underline{P}(\underline{y}_i) = h(\underline{y}_i)$, *and* $\underline{P}(\underline{z}_i) = g(\underline{z}_i)$, $i = 1, 2, \ldots, m+1$ *and m respectively.*

For the case $r + n = 2m + 1$, *there exists two perfect splines* $\tilde{P}$ *and* $\underline{P}$ *maximally oscillating between* $h(x)$ *and* $g(x)$ *such that* $h(x) \leqq \tilde{P}(x)$, $P\text{-}(x) \leqq g(x)$ *on* $[0, 1]$ *and for* $\tilde{P}(x)$ *there exists*

$$0 \leqq \tilde{y}_1 < \tilde{z}_1 < \tilde{y}_2 < \tilde{z}_2 < \ldots < \tilde{y}_{m+1} < \tilde{z}_{m+1} \leqq 1$$

where $\tilde{P}(\tilde{y}_i) = h(\tilde{y}_i)$, $\tilde{P}(\tilde{z}_i) = g(\tilde{z}_i)$, $i = 1, 2, \ldots, m+1$, *hold while there exists contact points* $Y = \{\underline{y}_i\}_1^{m+1}$, $Z = \{\underline{z}_i\}_1^{m+1}$ *for P to the curves h and g respectively, where the points of Y and Z interlace such that*

$$0 \leqq \underline{z}_1 < \underline{y}_1 < \cdots < \underline{z}_{m+1} < \underline{y}_{m+1} \leqq 1.$$

Several important corollaries ensue out of Theorem 2. We highlight two of them.

COROLLARY 2. *Let Q be any positive polynomial on* $[0,1]$ *of degree* $\leqq n - 1$. *Let r be a given positive integer. Then the unique representation*

$$Q(x) = \tilde{P}(x) + \underline{P}(x), \quad \textit{for all } x, \tag{5}$$

prevails where $\tilde{P}$ *and* $\underline{P}$ *are the maximally oscillating splines of Theorem* 2 (*uniquely determined*) *associated with* $g(x) = Q(x)$ *and* $h(x) \equiv 0$.

Specializing, Theorem 2 to the case of $h(x) \equiv -1$ and $g(x) \equiv 1$ we obtain

THEOREM 3. *For given n and r there exists a unique perfect spline* $P_{n,r}(x)$ *of degree n with r knots oscillating maximally over the interval* $[0, 1]$ *between* 1 *and* -1 *in the sense of Theorem* 2 *and normalized by the condition* $P_{n,r}(0) = 1$. *It follows that* $P_{n,r}(1) = (-1)^{n+r}$.

The construction of the special $P_{n,r}$ can be achieved without need of the full force of Theorems 1 and 2 by appropriate application of the Brouwer fixed point theorem. The case of Theorem 3 was announced by Schoenberg and Cavaretta [**4**]. The function $P_{n,r}$ also features prominently in the work of Tihomirov [**5**].

Designate $\rho_{n,r}$ as the constant value

$$\|P_{n,r}^{(n)}\|_\infty = \rho_{n,r}^{-1}(1) = P_{n,r}^{(n)}(1).$$

It is usually more convenient to normalize the nth derivative to 1. Accordingly we have

$$Q_{n,r}(x) = \rho_{n,r} P_{n,r}(x) \tag{6}$$

so that

$$\|Q_{n,r}\|_\infty = \rho_{n,r} = Q_{n,r}(0) = (-1)^{n+r} Q_{n,r}(1) \quad \text{and} \quad \|Q_{n,r}^{(n)}\|_\infty = 1.$$

It can be proved that $\rho_{n,0} > \rho_{n,1} > \rho_{n,2} > \ldots > \rho_{n,r} \to 0$ as $r \to \infty$.

A more basic version of Theorem 3 is stated below whose proof falls back on the interpolation property of perfect splines enunciated in Theorem 1 in conjunction with some further topological analysis.

THEOREM 4. *Let ρ be prescribed satisfying $0 < \rho < \infty$. There exists a unique perfect spline $Q(x;\rho)$ of degree n satisfying $|d^nQ(x;\rho)/dx^n| \equiv 1$, $|Q(x;\rho)| \leqq \rho$ on $[0,1]$ and $Q(0;\rho) = \rho$ such that, if $\rho_{n,r-1} > \rho > \rho_{n,r}$, $Q(x;\rho)$ involves exactly r knots and oscillates $n+r-1$ or $n+r$ times between ρ and $-\rho$, as x traverses $[0,1]$. Also, $Q(x;\rho)$ vanishes $n+r$ times on $[0,\infty)$ displaying at most one zero outside $[0,1]$. When $\rho = \rho_{n,r}$, of course, $Q(x,\rho) = Q_{n,r}(x)$.*

If γ_{n+r} is the largest zero of $Q'_{n,r}(x)$ then, for $\rho_{n,r} \leqq \rho \leqq [\gamma_{n+r}]^n$ (it can be shown that $[\gamma_{n+r}]^n < \rho_{n,r-1}$; by definition $\rho_{n,-1} = +\infty$), we have

$$Q(x;\rho) = \frac{\rho}{\rho_{n,r}} Q_{n,r}((\rho_{n,r}/\rho)^{1/n} x), \qquad r = 0,1,2,3,\ldots.$$

On the range $[\gamma_{n+r}]^n < \rho < \rho_{n,r-1}$, $Q(x;\rho)$ vanishes once on $(1,\infty]$.

The perfect splines $Q(x;\rho)$ enjoy several remarkable optimum properties. We highlight three typical situations.

(a) Consider the class $\mathscr{C}(\rho)$ of all $f \in W_\infty^{(n)}$ defined on $[0,1]$ obeying the restrictions

$$(7) \qquad \|f^{(n)}\|_\infty \leqq 1 \quad \text{and} \quad \|f\| \leqq \rho, \qquad \rho > 0 \text{ fixed}.$$

Then, for any $1 \leqq \nu \leqq n-1$,

$$(8) \qquad \max_{f\in\mathscr{C}(\rho)} |f^{(\nu)}(0)| = |Q^{(\nu)}(0;\rho)|, \qquad \max_{f\in\mathscr{C}(\rho)} |f^{(\nu)}(1)| = |Q^{(\nu)}(1;\rho)|.$$

(b) Let $0 < x^* < 1$. Consider the class $\mathscr{C}(\rho;x^*)$ of f in $\mathscr{C}(\rho)$ with the further property that $f(x^*) = Q(x^*;\rho)$. Suppose $Q'(x^*;\rho) > 0$. (If negative, replace max in (9) by min.) We have

$$(9) \qquad \max_{f\in\mathscr{C}(\rho;x^*)} f'(x^*) = Q'(x^*;\rho).$$

(c) Let $n + r = 2m$. Suppose $f \in W_\infty^{(n)}$ on $[0,b]$, $b > 1$, and satisfies the constraints

$$(10) \qquad \|f^{(n)}\|_\infty \leqq 1 \quad \text{on } [0,b]$$

but

$$\|f\| \leqq \rho_{n,r} \quad \text{on } [0,1].$$

Call this class of functions $\mathscr{C}_{n,r}[0,b]$. Then, we have

(11) $$\max_{f \in \mathscr{C}_{n,r}[0,b]} f(b) = Q_{n,r}(b).$$

It is sometimes useful to renormalize $Q_{n,r}$ as follows

(12) $$R_{n,r}(x) = Q_{n,r}(x\, \rho_{n,r}^{1/n})/\rho_{n,r}.$$

Clearly

$$|R_{n,r}^{(n)}(x)| \equiv 1 \quad \text{and} \quad |R_{n,r}(x)| \leqq 1 \text{ on } 0 \leqq x \leqq [\rho_{n,r}]^{-1/n}.$$

Letting $r \to \infty$, $E_n(x) = \lim_{r\to\infty} R_{n,r}(x)$ exists with convergence occurring uniformly on bounded intervals and we have

$$|E_n^{(n)}(x)| \equiv 1, \qquad -\infty \leqq x < \infty,$$
$$|E_n(x)| \leqq 1, \qquad 0 \leqq x < \infty.$$

Schoenberg and Cavaretta call $E_n(x)$ the nth order one-sided Euler spline. These authors proved that for the class $\mathscr{D}_n$ consisting of all functions f satisfying $|f^{(n)}(x)| \leqq 1$ and $|f(x)| \leqq 1$ on $(0, \infty)$ we have

$$\sup_{f\in\mathscr{D}_n} \sup_{0\leqq x<\infty} |f^{(\nu)}(x)| = \sup_{0\leqq x<\infty} |E_n^{(\nu)}(x)|, \qquad 0 < \nu < n.$$

This conclusion refines a classical inequality originating with Landau (and dealt with by many authors) concerned with securing estimates of $\|f^{(\nu)}\|_\infty$ in terms of the norms of $\|f\|_\infty$ and $\|f^{(n)}\|_\infty$.

Perfect splines occur also in calculating the nth diameter of the class $W_r = \{f; |f^{(r-1)}(x) - f^{(r-1)}(x')| \leqq |x - x'|\}$ (see Tihomirov [**5**]).

The theorems enunciated in this report and other enticing properties of perfect splines will be elaborated elsewhere.

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