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INJECTIVE MODULES AND CLASSICAL LOCALIZATION IN NOETHERIAN RINGS

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One of the main problems in the growing theory of noncommutative Noetherian rings can be loosely stated thus: If $\mathfrak{p}$ is a prime ideal of a Noetherian ring R, what should one mean by the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$? When does $R_{\mathfrak{p}}$ exist and when is it nice? This problem has been considered by Goldie [**1**] and by Lambek and Michler [**5**]. In this note, we indicate a new approach to this problem and some of its advantages. We also introduce the concept of a left exact biradical for a ring, which may be of independent interest. Details will appear elsewhere.

As usual, a ring is Noetherian if it has the ascending chain condition on right ideals as well as left ideals. A subset of a ring is an Ore set if it is right Ore as well as left Ore. We refer the reader to [**9**] for all unexplained terminology and results concerning left exact radicals.

Let R be a ring. The complete lattice of all left exact radicals for mod-R (resp. R-mod) is denoted as $\boldsymbol{K}_r$ (resp. $\boldsymbol{K}_l$). If $\mathscr{D}$ is a multiplicatively closed subset of R, $\rho_{\mathscr{D}} \in \boldsymbol{K}_r$ and $\lambda_{\mathscr{D}} \in \boldsymbol{K}_l$ are defined as follows: For each $M \in$ mod-R (resp. $M \in R$-mod), $\rho_{\mathscr{D}}(M)$ (resp. $\lambda_{\mathscr{D}}(M)$) is the largest submodule of M, each element of which is annihilated by some element of $\mathscr{D}$. If $\mathfrak{a}$ is an ideal of R, we define $\rho_{\mathfrak{a}}^{\#}$ as $\sup\{\rho \in \boldsymbol{K}_r | \rho(R/\mathfrak{a}) = 0\}$ and $\lambda_{\mathfrak{a}}^{\#}$ as $\sup\{\lambda \in \boldsymbol{K}_l | \lambda(R/\mathfrak{a}) = 0\}$. The multiplicatively closed set $\{r \in R | [r + \mathfrak{a}]$ is regular in $R/\mathfrak{a}\}$ is denoted as $\mathscr{C}(\mathfrak{a})$.

THEOREM 1 (*cf.* [**5**]). *If* $\mathfrak{s}$ *is a semiprime ideal in a right Noetherian ring then* $\rho_{\mathfrak{s}}^{\#} = \rho_{\mathscr{C}(\mathfrak{s})}$.

Matlis [**6**] has used localization to show that injective modules over a commutative Noetherian ring are nice. In the following two theorems, we establish an intimate connection between localizability and niceness of certain right injectives over a right Noetherian ring. Also see Theorems 7 and 8.

THEOREM 2. *Let* $\mathfrak{s}$ *be a semiprime ideal in a right Noetherian ring* R. *Then the following four conditions are equivalent*:

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(1) *$\mathscr{C}(\mathfrak{s})$ is a right Ore set in R.*

(2) *There exists a right Ore set $\mathscr{C}$ in R such that $\rho_{\mathscr{C}} = \rho_{\mathfrak{s}}^{\#}$.*

(3) *Let $\mathscr{D}$ be any multiplicatively closed subset of R such that $\mathscr{D} \subseteq \mathscr{C}(\mathfrak{s})$ and $\rho_{\mathscr{D}} = \rho_{\mathfrak{s}}^{\#}$. Then $\mathscr{D}$ is right Ore in R.*

(4) *Let N be any right R-module which is R-isomorphic with a uniform right ideal of the ring $R/\mathfrak{s}$. Let M_R be any essential extension of N_R such that M/N is $\rho_{\mathfrak{s}}^{\#}$-torsion. Then $\mathfrak{s} \subseteq$ ann M.*

THEOREM 3. *Let $\mathfrak{s}$ be a semiprime ideal in a right Noetherian ring R and let $\bar{R} = R/\rho_{\mathfrak{s}}^{\#}(R)$. Assume that $\mathscr{C}(\mathfrak{s})$ is a right Ore set in R. Then,*

(1) *$\rho_{\mathfrak{s}}^{\#}(R) \subseteq \mathfrak{s}$ and $\bar{\mathfrak{s}} = \mathfrak{s}/\rho_{\mathfrak{s}}^{\#}(R)$ is a semiprime ideal in the right Noetherian ring $\bar{R}$. The image of $\mathscr{C}(\mathfrak{s})$ in $\bar{R}$ is $\mathscr{C}(\bar{\mathfrak{s}})$ which is a right Ore set of regular elements of $\bar{R}$. If R is a semiprime ring, so is $\bar{R}$.*

(2) *Let $R_{\mathfrak{s}}$ denote the classical right quotient ring of $\bar{R}$ with respect to $\mathscr{C}(\bar{\mathfrak{s}})$. Then $R_{\mathfrak{s}}$ is a semilocal right Noetherian ring with $J(R_{\mathfrak{s}}) = \bar{\mathfrak{s}}R_{\mathfrak{s}}$. The classical total right quotient ring of $\bar{R}/\bar{\mathfrak{s}}$ is isomorphic with $R_{\mathfrak{s}}/J(R_{\mathfrak{s}})$.*

(3) *The injective hull of $R/\mathfrak{s}$ in* mod-*R is R-isomorphic with the injective hull of $\bar{R}/\bar{\mathfrak{s}}$ in* mod-*$\bar{R}$ which, in turn, is R-isomorphic with the injective hull of $R_{\mathfrak{s}}/J(R_{\mathfrak{s}})$ in* mod-*$R_{\mathfrak{s}}$.*

The following example suggests that, in an attempt to localize a Noetherian ring R at a prime ideal $\mathfrak{p}$, one should *not* overemphasize the set $\mathscr{C}(\mathfrak{p})$. Let $n > 1$ be a positive integer and let R be the subring of $M_n(\boldsymbol{Z})$, consisting of all those matrices in which all the entries below the main diagonal belong to $2\boldsymbol{Z}$. Let $\mathfrak{p}_i$, $1 \leqq i \leqq n$, be the maximal ideal of R consisting of all those matrices in which the (i, i)th entry belongs to $2\boldsymbol{Z}$. One can easily see that, in mod-R, the sequence

$$(*) \qquad 0 \to \frac{\mathfrak{p}_i \cap \mathfrak{p}_{i+1}}{\mathfrak{p}_i\mathfrak{p}_{i+1}} \to \frac{\mathfrak{p}_{i+1}}{\mathfrak{p}_i\mathfrak{p}_{i+1}} \to \frac{\mathfrak{p}_{i+1}}{\mathfrak{p}_i \cap \mathfrak{p}_{i+1}} \to 0$$

is exact and nonsplit and that

$$\frac{\mathfrak{p}_i \cap \mathfrak{p}_{i+1}}{\mathfrak{p}_i\mathfrak{p}_{i+1}} \cong \frac{R}{\mathfrak{p}_{i+1}}, \qquad \frac{\mathfrak{p}_{i+1}}{\mathfrak{p}_i \cap \mathfrak{p}_{i+1}} \cong \frac{R}{\mathfrak{p}_i},$$

the indexing being modulo n. What should the localization of R at $\mathfrak{p}_1$ be? In view of condition (4) of Theorem 2, the sequence (*) with $i = n$ indicates a "tie" of $\mathfrak{p}_1$ with $\mathfrak{p}_n$ and this obvious obstacle prevents $\mathscr{C}(\mathfrak{p}_1)$ from being a right Ore set in R. Condition (4) of Theorem 2 also suggests a remedy viz., try $\mathscr{C}(\mathfrak{p}_1 \cap \mathfrak{p}_n)$. However, if $n > 2$ then the sequence (*) with $i = n - 1$ indicates a tie of $\mathfrak{p}_1$ with $\mathfrak{p}_{n-1}$ via $\mathfrak{p}_n$ and this prevents $\mathscr{C}(\mathfrak{p}_1 \cap \mathfrak{p}_n)$ from being a right Ore set in R. (Note: $\mathrm{Ext}_R^1(R/\mathfrak{p}_{n-1}, R/\mathfrak{p}_1) = (0)$.) In this way, one can see that if $\mathfrak{a}$ is any ideal of R such that $\bigcap_{i=1}^n \mathfrak{p}_i \subsetneqq \mathfrak{a} \subseteq \mathfrak{p}_1$ then there is an obvious obstacle which prevents $\mathscr{C}(\mathfrak{a})$ from being a right or left

Ore set in R. There is nothing obvious to prevent $\mathscr{C} = \mathscr{C}(\bigcap_{i=1}^{n} \mathfrak{p}_i)$ from being Ore. Indeed, it can be shown that $\mathscr{C}$ is an Ore set of regular elements of R and that the localization of R at $\mathscr{C}$ is the usual localization of the Z-order R at the prime 2 in $\mathbf{Z}$.

This example suggests that, given a prime ideal $\mathfrak{p}$ in a Noetherian ring R, one should seek a semiprime ideal $\gamma(\mathfrak{p})$ such that the associated prime ideals of $\gamma(\mathfrak{p})$ are precisely those prime ideals which have a "tie" with $\mathfrak{p}$ and then examine whether $\mathscr{C}(\gamma(\mathfrak{p}))$ is right Ore; if this set fails then $\mathfrak{p}$ is beyond first aid. In the context of HNP-rings with enough invertibles, a localization along these lines was developed by the present author [**3**]. Compared to the HNPR case, the "ties" between prime ideals in an arbitrary Noetherian ring are far from visible. To get an idea about these ties and get a candidate for $\gamma(\mathfrak{p})$, we have to introduce the notion of a "left exact biradical for a ring".

A *left exact biradical* for a ring R is an ordered pair $(\lambda, \rho) \in \mathbf{K}_l \times \mathbf{K}_r$ such that $\lambda(R/\mathfrak{t}) = \rho(R/\mathfrak{t})$ for every ideal $\mathfrak{t}$ of R. The partial order on the set $\mathbf{K}$ of all left exact biradicals for R is defined by restricting the product partial order on $\mathbf{K}_l \times \mathbf{K}_r$. It turns out that $(\mathbf{K}, \leqq)$ is a complete lattice. If $\mathfrak{a}$ is an ideal of R, we define $(\lambda_\mathfrak{a}, \rho_\mathfrak{a})$ as $\sup\{(\lambda, \rho) \in \mathbf{K} | \rho(R/\mathfrak{a}) = 0\}$. Clearly, $\rho_\mathfrak{a} \leqq \rho_\mathfrak{a}^{\#}$ and $\lambda_\mathfrak{a} \leqq \lambda_\mathfrak{a}^{\#}$; however, these inequalities may be strict. The particularly interesting case when $\mathfrak{a} = 0$ will be dealt with elsewhere.

If R is a commutative ring then there is an obvious bijection between $\mathbf{K}$ and $\mathbf{K}_l = \mathbf{K}_r$. If R is a semiprimary ring then there is a bijection between $\mathbf{K}$ and the set of central idempotents of R. If $\mathscr{D}$ is an Ore set in a Noetherian ring R, it can be shown that $(\lambda_\mathscr{D}, \rho_\mathscr{D}) \in \mathbf{K}$.

Henceforth, R will denote a Noetherian ring, $\mathbf{P}(R)$ will denote the set of all prime ideals of R and $\mathfrak{s}$ will denote a semiprime ideal of R. Set $\Gamma_0(\mathfrak{s}) = \{\mathfrak{p} \in \mathbf{P}(R) | \rho_\mathfrak{s}(R/\mathfrak{p}) = 0\}$. Let $\Gamma(\mathfrak{s})$ be the set of all those prime ideals of R which are maximal in the set $\Gamma_0(\mathfrak{s})$. The set $\Gamma(\mathfrak{s})$ is our candidate for the set of all those prime ideals of R which are "tied" to some prime ideal associated with $\mathfrak{s}$.

THEOREM 4. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the prime ideals associated with a semiprime ideal $\mathfrak{s}$ of a Noetherian ring R. Let $\Gamma(\mathfrak{s}) \subseteq \Gamma \subseteq \Gamma_0(\mathfrak{s})$. Then $\Gamma_0(\mathfrak{s}) = \bigcup_{i=1}^{n} \Gamma_0(\mathfrak{p}_i)$, $\Gamma(\mathfrak{s}) \subseteq \bigcup_{i=1}^{n} \Gamma(\mathfrak{p}_i)$ and $\rho_\mathfrak{s} = \inf_{1 \leqq i \leqq n} \rho_{\mathfrak{p}_i} = \inf\{\rho_\mathfrak{p}^{\#} : \mathfrak{p} \in \Gamma\}$.*

With appropriate definitions, it can be shown that $(\lambda_\mathfrak{s}, \rho_\mathfrak{s})$ is a prime (resp. semiprime) in $\mathbf{K}$ if $\mathfrak{s}$ is a prime (resp. semiprime) ideal of R (cf. [**2**]).

Let $\mathfrak{m}, \mathfrak{n} \in \mathbf{P}(R)$. We use the symbol $\mathfrak{m} \rightsquigarrow \mathfrak{n}$ to signify that there exist ideals $\mathfrak{a} \subsetneqq \mathfrak{b}$ in R such that $\mathfrak{m}\mathfrak{b} + \mathfrak{b}\mathfrak{n} \subseteq \mathfrak{a}$ and $\mathfrak{b}/\mathfrak{a}$ is nonsingular in $(R/\mathfrak{m})$-mod as well as mod-$(R/\mathfrak{n})$. If there exists a finite sequence $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ such that, for $1 \leqq i \leqq k - 1$, either $\mathfrak{m}_i \rightsquigarrow \mathfrak{m}_{i+1}$ or $\mathfrak{m}_{i+1} \rightsquigarrow \mathfrak{m}_i$ then we set $\mathfrak{m}_1 \sim \mathfrak{m}_k$. If $\mathfrak{p} \in \mathbf{P}(R)$, let $\Omega(\mathfrak{p}) = \{\mathfrak{q} \in \mathbf{P}(R) | \mathfrak{p} \sim \mathfrak{q}\}$. In several cases, it can be shown that $\Gamma(\mathfrak{p}) = \Omega(\mathfrak{p})$. In general, we have

THEOREM 5. *If* $\mathfrak{p}$ *is a prime ideal in a Noetherian ring* R *then* $\Gamma_0(\mathfrak{p}) = \bigcup\{\Omega(\mathfrak{m}) : \mathfrak{m} \in \Gamma_0(\mathfrak{p})\}$.

If the set $\Gamma(\mathfrak{s})$ is finite, $\mathfrak{s}$ is called a *nondegenerate* semiprime ideal of R. For a nondegenerate $\mathfrak{s}$, we set $\gamma(\mathfrak{s}) = \bigcap\{\mathfrak{p} \in \Gamma(\mathfrak{s})\}$. If $\Gamma(\mathfrak{s})$ is precisely the set of prime ideals associated with $\mathfrak{s}$ then $\mathfrak{s}$ is said to be a *stable* semiprime ideal of R. It can be shown that a semiprime ideal $\mathfrak{s}$ is stable iff $\rho_\mathfrak{s} = \rho_\mathfrak{s}^\# = \rho_{\mathscr{C}(\mathfrak{s})}$ iff $\lambda_\mathfrak{s} = \lambda_\mathfrak{s}^\# = \lambda_{\mathscr{C}(\mathfrak{s})}$. In the example given above, $\gamma(\mathfrak{p}_1) = \bigcap_{i=1}^n \mathfrak{p}_i$ and it is stable.

We *conjecture* that all semiprime ideals in a Noetherian ring are nondegenerate and all but a finite number of them are stable.

THEOREM 6. *Let* $\mathfrak{s}$ *be a nondegenerate semiprime ideal in a Noetherian ring* R. *Then* $\gamma(\mathfrak{s})$ *is a stable semiprime ideal of* R, $\Gamma(\gamma(\mathfrak{s})) = \Gamma(\mathfrak{s})$ *and* $(\lambda_\mathfrak{s}, \rho_\mathfrak{s}) = (\lambda_{\gamma(\mathfrak{s})}, \rho_{\gamma(\mathfrak{s})}) = (\lambda_{\mathscr{C}\gamma(\mathfrak{s})}, \rho_{\mathscr{C}\gamma(\mathfrak{s})})$. *If* $\mathfrak{a}$ *is any stable semiprime ideal of* R *such that* $(\lambda_\mathfrak{s}, \rho_\mathfrak{s}) = (\lambda_\mathfrak{a}, \rho_\mathfrak{a})$ *then* $\mathfrak{a} = \gamma(\mathfrak{s})$. *If* $\mathscr{D}$ *is any Ore set in* R *contained in* $\mathscr{C}(\mathfrak{s})$ *then* $\mathscr{D} \subseteq \mathscr{C}(\gamma(\mathfrak{s}))$.

A nondegenerate semiprime ideal $\mathfrak{s}$ is said to be *classical* if $\mathscr{C}(\gamma(\mathfrak{s}))$ is an Ore set in R. Theorem 6 implies that if a prime ideal $\mathfrak{p}$ is classical in Goldie's sense [**1**] and if the intersection of the symbolic powers of $\mathfrak{p}$ is contained in $\rho_\mathfrak{p}(R)$ then $\mathfrak{p}$ is stable and classical in our sense.

Let $\mathfrak{s}$ be a nondegenerate semiprime ideal in a Noetherian ring R and let $\mathscr{D}$ be a one-sided Ore set in R such that $\mathscr{C}(\gamma(\mathfrak{s})) \subseteq \mathscr{D} \subseteq \mathscr{C}(\mathfrak{s})$. Is $\mathscr{D}$ necessarily a two-sided Ore set in R? The available information suggests that the answer should be in the affirmative.

We now indicate some applications of our approach to localization. Recall that a prime Noetherian ring is bounded if every essential one-sided ideal contains a nonzero two-sided ideal. A Noetherian ring R is *fully bounded* if $R/\mathfrak{p}$ is bounded for every $\mathfrak{p} \in \boldsymbol{P}(R)$. It is well known that a Noetherian ring R is fully bounded if R is finitely generated as a module over its centre; in such a ring R, it can be shown that every semiprime ideal is classical.

THEOREM 7. *If* $\mathfrak{s}$ *is a nondegenerate semiprime ideal in a fully bounded Noetherian ring* R *then* $\mathfrak{s}$ *is classical and the classical ring of quotients of* R *with respect to the Ore set* $\mathscr{C}(\gamma(\mathfrak{s}))$ *is a semilocal fully bounded Noetherian ring.*

THEOREM 8. *Let* R *be a fully bounded Noetherian ring. Then* $\bigcap_{n=1}^\infty J^n(R) = (0)$. *If* E *is the injective hull of a simple right or left* R-*module then any finitely generated submodule of* E *has finite length.*

Assume that R *is semilocal as well. Let* $\mathfrak{m}$ *be a maximal ideal of* R. *Then*

$\Gamma(\mathfrak{m})$ consists of those maximal ideals $\mathfrak{n}$ of R which have the following property: There exists a finite sequence $\mathfrak{m} = \mathfrak{m}_1, \ldots, \mathfrak{m}_k = \mathfrak{n}$ of maximal ideals of R such that $(\mathfrak{m}_i\mathfrak{m}_{i+1}) \cap (\mathfrak{m}_{i+1}\mathfrak{m}_i) \neq \mathfrak{m}_i \cap \mathfrak{m}_{i+1}$ for $1 \leqq i \leqq k-1$. In particular, $\Gamma(\mathfrak{m}) = \Omega(\mathfrak{m})$.

Theorems 2, 3, 7 and 8 show that a substantial portion of the well-known work of Matlis [**6**] on injectives over commutative Noetherian rings holds over fully bounded Noetherian rings. The finiteness assertions proved by Matlis can be obtained by imposing a suitable polynomial identity (cf. [**8**]).

Recall that a semiprime Noetherian ring R has Krull dimension one iff R/L is of finite length for every essential one-sided ideal L of R and R is nonsemisimple.

THEOREM 9. *Let R be a semiprime Noetherian ring of Krull dimension one. If a semiprime ideal $\mathfrak{s}$ of R contains an invertible ideal of R then $\mathfrak{s}$ is classical and $\gamma(\mathfrak{s})$ is the prime radical of any invertible ideal of R which is maximal among those contained in $\mathfrak{s}$. The classical quotient ring of R with respect to $\mathscr{C}(\gamma(\mathfrak{s}))$ is a fully bounded semilocal semiprime Noetherian ring of Krull dimension one. Any right or left R-module M of finite length can be uniquely decomposed as $M = K \oplus L$ where every composition factor of K is annihilated by $\gamma(\mathfrak{s})$ and no composition factor of L is annihilated by $\gamma(\mathfrak{s})$.*

The above theorem shows that the usual localization of classical orders over commutative Dedekind domains [**7**] and the localization in HNPR developed in [**3**], [**4**] are special cases of our localization.

Let R be a semiprime Noetherian ring with total quotient ring Q. Let $\mathfrak{s}$ be a semiprime ideal of R such that $\rho_{\mathfrak{s}}(R) = (0)$; this condition is trivially satisfied if R is a prime ring. The rings of quotients $Q_{\rho_{\mathfrak{s}}}(R)$ and $Q_{\lambda_{\mathfrak{s}}}(R)$ can be realized as subrings of Q. The subring $B_{\mathfrak{s}}(R) = Q_{\rho_{\mathfrak{s}}}(R) \cap Q_{\lambda_{\mathfrak{s}}}(R)$ of Q may be an appropriate candidate for the localization of R at $\mathfrak{s}$ even when $\mathfrak{s}$ is not classical. This construction can be generalized but, at present, we do not know whether the ring $B_{\mathfrak{s}}(R)$ is of any interest in connection with R.

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