BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

# SPACES OF EQUIVARIANT SELF-EQUIVALENCES OF SPHERES

BY J. C. BECKER[1] AND R. E. SCHULTZ[2]

Communicated by William Browder, July 13, 1972

ABSTRACT. Let $F(S^m)$ denote the identity component of the space of homotopy self-equivalences of $S^m$ and let $F = \text{inj lim}_m F(S^m)$. This paper studies the homotopy properties of certain equivariant analogs of the infinite loop space $F$.

1. **Introduction.** Let $G$ be a compact Lie group and let $W$ be a free, finite dimensional, real $G$-module equipped with a $G$-invariant metric. Let $S(W)$ be the unit sphere of $W$ and denote by $F(W)$ the identity component of the space of equivariant self-equivalences of $S(W)$ with the compact-open topology.

If $V$ and $W$ are free $G$-modules as above, then $V \oplus W$ is also a free $G$-module. Since $S(V \oplus W)$ is equivariantly homeomorphic to the join of $S(V)$ and $S(W)$, there is a continuous inclusion of $F(V)$ into $F(V \oplus W)$ defined by taking joins with the identity on $S(W)$. In particular, if $kW$ denotes the direct sum of $k$ copies of $W$, there is an inclusion of $F(kW)$ in $F((k+1)W)$. Define

$$F_G = \text{inj lim}_k F(kW). \tag{1.1}$$

If $G$ is the trivial group then $F_G = F$ is a familiar and widely studied object. An important aspect of this space is the existence of two infinite loop space structures, one induced by composition multiplication, the other induced by a canonical homotopy equivalence from $F$ to the identity component of inj $\lim_m \Omega^m(S^m)$. One can show that $F_G$ also has an infinite loop space structure induced by composition multiplication. Our results generalize to $F_G$ the second infinite loop space structure on $F$.

Let $BG$ denote a classifying space for $G$, let $\mathfrak{g}$ be the Lie algebra of $G$ and let $G$ act on $\mathfrak{g}$ via the adjoint representation. The balanced product of $EG$ and $\mathfrak{g}$ is a vector bundle over $BG$ that we shall call $\zeta$. Let $BG^\zeta$ denote its Thom space.

THEOREM 1. *On the category of connected finite* CW-*complexes there is a natural equivalence of homotopy functors*

---

*AMS (MOS) subject classifications* (1970). Primary 55D10, 55D35; Secondary 55E45.

*Key words and phrases.* Free linear representation, equivariant self-equivalence, infinite loop space, transfer map, ex-space.

[1] Partially supported by NSF Grant GP-34197.

[2] Partially supported by NSF Grant GP-19530.

$$\lambda_G : [\ ; F_G] \to \{\ ; BG^\zeta\}$$

Here $\{A; B\}$ denotes the homotopy classes of pointed $S$-maps from $A$ to $B$.

If $Y$ is a pointed space, let $Q_0(Y)$ be the identity component of inj $\lim_k \Omega^k S^k(Y)$. The exponential law provides a natural equivalence from $\{\ ; Y\}$ to $[\ ; Q_0(Y)]$ on the category of connected finite CW-complexes. Combining this with Theorem 1, we obtain the following result.

**THEOREM** 2.[3] *The space $F_G$ is homotopy equivalent to $Q_0(BG^\zeta)$.*

If $G$ is the trivial group, Theorem 1 reduces to the usual equivalence

$$\lambda[\ ;F] \to \{\ ;S^0\} \tag{1.2}$$

given by sending $f: X \to F(S^n)$ to the map $h(f): X * S^n \to S^{n+1}$ obtained by applying the Hopf construction to the adjoint of $f$. If $G$ is a finite group, then $\zeta$ is 0-dimensional and $BG^\zeta = BG^+$, the disjoint union of $BG$ with a point. In this case $F_G$ has the homotopy type of $Q_0(BG) \times Q_0(S^0)$. The only compact Lie groups of positive dimension that act freely on spheres are $S^1$, $S^3$, and $N(S^1)$, the normalizer of $S^1$ in $S^3$ [**5**]. If $G = S^1$ then $\zeta$ is a trivial line bundle and in this case $F_G$ has the homotopy type of $Q_0(CP^\infty) \times Q_0(S^1)$. If $G = S^3$, then $BG^\zeta$ is an infinite dimensional quasi-projective space as defined by James (see [**2**, Proposition (5.3)]).

2. **Naturality properties.** Let $G$ be as above and let $H$ be a closed subgroup of $G$. Then we may take $BH$ to be $EG/H$, and the canonical map from $BH$ to $BG$ to be the projection. Techniques of J. M. Boardman [**4**] imply the existence of a "wrong way" map (in the stable homotopy category)

$$\tau: BG^{\zeta(G)} \to BH^{\zeta(H)}. \tag{2.1}$$

If $G$ and $H$ are finite, $\tau$ agrees with the transfer defined in [**9**]. Let

$$\rho: F_G \to F_H \tag{2.2}$$

denote the natural forgetful map.

**THEOREM** 3. *The following diagram is commutative*

$$\begin{array}{ccc} [\ ;F_G] & \xrightarrow{\rho_*} & [\ ;F_H] \\ \downarrow \lambda_G & & \downarrow \lambda_H \\ \{\ ;BG^{\zeta(G)}\} & \xrightarrow{\tau_*} & \{\ ;BH^{\zeta(H)}\}. \end{array}$$

If $G$ and $H$ are finite, the map

[3] ADDED IN PROOF. Spaces related to $F_G$ have been studied by G. Segal [**12**] using bordism techniques. Theorem 2 is similar to Proposition 4 of [**12**].

$$p_*^+ : \{\ ;BH^+\} \to \{\ ;BG^+\} \tag{2.3}$$

has a geometrical interpretation in terms of a transfer map $t: F_H \to F_G$; details will appear elsewhere.

3. **Applications.** The above results are useful in describing the image of

$$\rho_* : \pi_*(F_G) \to \pi_*(F_H).$$

For example, a theorem of D. S. Kahn and S. B. Priddy [**8**] implies that the transfer

$$\tau : \Sigma_n(RP^{\infty +}) \to \Sigma_n(S^0), \qquad n > 0,$$

is surjective. Hence we have the following.

THEOREM 4. *The forgetful map* $\rho_* : \pi_*(F_{Z_2}) \to \pi_*(F)$ *is surjective.*

On the other hand we have the following result.

THEOREM 5. *Let $k$ be a positive integer, let $\sigma_k \in \pi_{8k-1}(F)$ generate the image of $J$, and let $\mu_k \in \pi_{8k+1}(F)$ be an Adams-Barratt element* [**1**]. *Then neither $\sigma_k$ nor $\mu_k$ is in the image of* $\rho_* : \pi_*(F_{S^1}) \to \pi_*(F)$.

Geometrical applications of the result on $\mu_k$ will be given in [**10**].

4. **Spaces over *B*.** Fix a CW-complex $B$ and let $\mathscr{C}(B)$ denote the category having objects $\xi = (E_\xi, B, p_\xi, \Delta_\xi)$ where $p_\xi : E_\xi \to B$ is a fiber bundle and $\Delta_\xi$ is a cross section to $p_\xi$. We assume that $\xi$ is admissible in the sense of [**3**]. In the terminology of James [**7**], $\xi$ is an ex-space of $B$. The set $[\xi, \xi']$ of maps in $\mathscr{C}(B)$ is the set of homotopy classes of fiber and cross section preserving maps $E_\xi \to E_{\xi'}$. The category $\mathscr{C}(B)$ is a natural extension of the category of pointed spaces, and much of the homotopy theory of pointed spaces can be extended to $\mathscr{C}(B)$. For detailed accounts see [**3**], [**6**], [**7**].

Let $\xi \wedge \alpha$ denote the fiberwise reduced join of $\xi$ and $\alpha$ and define

$$\sigma : [\xi ; \xi'] \to [\xi \wedge \alpha ; \xi' \wedge \alpha] \tag{4.1}$$

by $f \to f \wedge 1$. We then have the following suspension theorem (compare [**6**, Theorem (7.4)]).

THEOREM 6. *Assume that $\alpha$ is a sphere bundle and the fiber of $\xi'$ is $(n-1)$-connected. Then $\sigma$ is injective if $E_\xi$ is $(2n-1)$-coconnected and surjective if $E_\xi$ is $2n$-coconnected.*

Let $T(\xi) = E_\xi/\Delta_\xi(B)$. If $X$ is a space with base point $x_0$ let $\dot{X}$ denote the object $(B \times X, B, p, \Delta)$ where $p(b, x) = b$ and $\Delta(b) = (b, x)$. Note that $T(X \wedge \xi) = X \wedge T(\xi)$. Observe also that the projection map $B \times X$

$\to X$ induces a one-one correspondence $[\xi; X] \to [T(\xi); X]$.

If $\beta$ is a vector bundle over $B$ let $\bar{\beta}$ denote the object of $\mathscr{C}(B)$ obtained by taking the fiberwise one point compactification of $E_\beta$ and letting $\Delta_{\bar{\beta}}$ be the cross section at infinity.

5. **Proof of Theorem 1.** Let $W$ be a free $G$-module of dimension $n$, let $M(W) = S(W)/G$ and let $\xi = (S(W) \times S(W)/G, M(W), p, \Delta)$ where $p[w, w'] = [w]$ and $\Delta[w] = [w, w]$. Suppose that $X$ is a finite connected complex and $\dim(X) < n - 2$. We have a bijection

$$\theta: [X; F(W)] \to [\dot{X}; \xi] \tag{5.1}$$

defined as follows: given $f: X \to F(W)$ define

$$\theta(f): M(W) \times X \to S(W) \times S(W)/G$$

by

$$\theta(f)([w], x) = [w, f(x)(w)].$$

If $M$ is a smooth manifold let $\tau(M)$ denote its tangent bundle. Let $\zeta$ denote the bundle with fiber $\mathfrak{g}$ associated with the principal bundle $S(W) \to M(W)$. We then have [**11**]

$$\xi \simeq \overline{\tau(S(W))/G} \simeq \overline{\tau(M(W)) \oplus \zeta}.$$

Making this identification (and abbreviating $\tau(M(W))$ to $\tau$) we have

$$\theta: [X; F(W)] \to [\dot{X}; \overline{\tau \oplus \zeta}]. \tag{5.2}$$

Now choose (a) an embedding $h: M(W) \subset R^s$ and (b) a monomorphism $\phi: \zeta \to B \times R^t$. Let $\nu$ denote the normal bundle determined by $h$ and $\zeta'$ the complementary bundle determined by $\phi$. From this data we obtain (a′) an equivalence $\psi: \overline{(\tau \oplus \zeta) \oplus (\nu \oplus \zeta')} \to S^{s+t}$ and (b′) a duality map $\mu: S^{s+t} \to T(\zeta) \wedge T(\nu \oplus \zeta')$.

Define

$$\kappa: [\dot{X}; \overline{\tau \oplus \zeta}] \to [X \wedge T(\nu \oplus \zeta'); S^{s+t}] \tag{5.3}$$

to be composition

$$[\dot{X}; \overline{\tau \oplus \zeta}] \xrightarrow{\sigma} [\dot{X} \wedge \overline{\nu \oplus \zeta'}; \overline{\tau \oplus \zeta \oplus \nu \oplus \zeta'}]$$

$$\xrightarrow{\psi_*} [\dot{X} \wedge \overline{\nu \oplus \zeta'}; \dot{S}^{s+t}] \to [X \wedge T(\nu \oplus \zeta'); S^{s+t}].$$

Since $\dim(X) < n - 2$, $\sigma$ and hence $\kappa$ is bijective.

The duality map $\mu$ defines a bijection

$$D_\mu: \{X \wedge T(\nu \oplus \zeta'); S^{s+t}\} \to \{X; T(\zeta)\}. \tag{5.4}$$

Since we are in the stable range we may define

$$\lambda_W : [X, F(W)] \to \{X; T(\zeta)\} \tag{5.5}$$

by $\lambda_W = D_\mu \kappa \theta$. It is easily seen that $\lambda_W$ is independent of the choice of $h$ and $\phi$. Moreover, if $V$ is a second free $G$-module, it is compatible with the inclusion $F(V) \to F(V \oplus W)$ in the obvious sense. Now $\lambda_G$ in Theorem 1 is defined to be inj $\lim_k \lambda_{kW}$.

## REFERENCES


**1.** J. F. Adams, *On the groups J(X)*. IV, Topology **5** (1966), 21–71. MR **33** #6628.

**2.** M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310. MR **24** #A1727.

**3.** J. C. Becker, *On the existence of $A_k$-structures on stable vector bundles*, Topology **9** (1970), 367–384. MR **42** #1135.

**4.** J. M. Boardman, *Stable homotopy theory*, University of Warwick, 1966 (mimeographed notes).

**5.** G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

**6.** I. M. James, *Bundles with special structure*. I, Ann. of Math. (2) **89** (1969), 359–390. MR **39** #4868.

**7.** ———, *Ex-homotopy theory*. I, Illinois J. Math. **15** (1971), 324–337.

**8.** D. S. Kahn and S. B. Priddy, *Applications of the transfer to stable homotopy theory* (to appear).

**9.** F. W. Roush, *Transfer in generalized cohomology theories*, Thesis, Princeton University, Princeton, N.J.

**10.** R. E. Schultz, *Homotopy sphere pairs admitting semifree differentiable actions* (to appear).

**11.** R. H. Szczarba, *On tangent bundles of fibre spaces and quotient spaces*, Amer. J. Math. **86** (1964), 685–697. MR **30** #2522.

**12.** G. B. Segal, *Equivariant stable homotopy theory*, Proc. Internat. Congress Math. (Nice, 1970), vol. 2, Gauthier-Villars, Paris, 1971, pp. 59–64.


DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, LAFAYETTE, INDIANA 47907