POSITIVE HARMONIC FUNCTIONS AND BIHARMONIC DEGENERACY

BY LEO SARIO AND CECILIA WANG

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The class $O_{HP}$ of Riemann surfaces or Riemannian manifolds which do not carry (nonconstant) positive harmonic functions is the smallest harmonically or analytically degenerate class. In particular, it is strictly contained in the classes $O_{HB}$ and $O_{HD}$ of Riemann surfaces or Riemannian manifolds without bounded or Dirichlet finite harmonic functions, and in the classes $O_{AB}$ and $O_{AD}$ of Riemann surfaces without bounded or Dirichlet finite analytic functions.

In the present paper we ask: Are there any relations between $O_{HP}$ and the classes $O_{H2B}$ and $O_{H2D}$ of Riemannian manifolds without bounded or Dirichlet finite nonharmonic biharmonic functions? We shall show that the answer is in the negative. Explicitly, if $O^{N}$ is a null class of $N$-dimensional manifolds, and $\bar{O}^{N}$ its complement, then all four classes

$$O_{HP}^{N} \cap O_{H2X}^{N}, \quad O_{HP}^{N} \cap \bar{O}_{H2X}^{N}, \quad \bar{O}_{HP}^{N} \cap O_{H2X}^{N}, \quad \bar{O}_{HP}^{N} \cap \bar{O}_{H2X}^{N}$$

are nonempty for both $X = B$ and $D$, and for any $N$. This independence of $N$ is of interest, as biharmonic degeneracy often fails to have this property. Typically, whereas the punctured Euclidean $N$-space is not an element of $O_{H2B}^{N}$ for $N = 2, 3$, it does belong to it for all $N \geq 4$ (Sario-Wang [6]).

Methodologically, we introduce in §1 a simple type of Riemannian manifold which, on account of its rectangular coordinates and nonconformal metric, is very versatile in classification problems.

1. We shall show

**Theorem 1.** $O_{HP}^{N} \cap \bar{O}_{H2B}^{N} \neq \emptyset$ for every $N$.

**Proof.** Consider the $N$-manifold, $N \geq 2$,

$$T = \{0 < x < \infty, 0 \leq y \leq 2\pi, 0 \leq z_{i} \leq 2\pi\} \quad \text{i = 1, \ldots, N - 2},$$

with $y = 0, y = 2\pi$ identified, and $z_{i} = 0, z_{i} = 2\pi$ also identified for every $i$. Endow $T$ with the metric


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To see that $T \in O_{HP}$ note that $h(x) \in H(T)$ if $\Delta h = -x^{-1}d(\Delta h)/dx = 0$, that is, $h = a \log x + b$ with constants $a, b$. Since $|h| \to \infty$ as $x \to 0$ or $\infty$, the harmonic measure of the ideal boundary of $T$ vanishes, and $T$ belongs to the class $O_G$ of parabolic manifolds. In view of $O_G \subset O_{HP}$ (see e.g. Sario-Nakai [4]), we have $T \in O_{HP}$.

An $H^2 B$-function on $T$ is $u = \sin 2y$. In fact,

$$\Delta u = -x^{-1} \partial(x^{-1} \cdot 2 \cos 2y)/\partial y = 4x^{-2} \sin 2y$$

and

$$\Delta^2 u = -4x^{-1} \left\{ \frac{\partial}{\partial x} [x \cdot (-2x^{-3}) \sin 2y] + \frac{\partial}{\partial y} [x^{-1} \cdot x^{-2} \cdot 2 \cos 2y] \right\} = 0.$$
\[ u_{nm} = -\frac{1}{2p_n} r^{p_n} \log r \cdot S_{nm}, \quad v_{nm} = -\frac{1}{2q_n} r^{q_n} \log r \cdot S_{nm}, \]
\[ \tau = -\frac{1}{6}(\log r)^3, \quad s = -\frac{1}{2}(\log r)^2. \]

Every \( u \in H^2(M) \) has an expansion
\[
u = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} u_{nm} + b_{nm} v_{nm}) + \alpha \tau + b s
+ \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm} h_{nm} + d_{nm} k_{nm}) + c \sigma + d
\]
on \( M \), with compact convergence implied by that of the expansion of \( h \).
For fixed \( (n, m) \),
\[
\int_S u \cdot S_{nm} \, d\omega = Ar^{p_n} \log r + Br^{q_n} \log r + Cr^{p_n} + Dr^{q_n},
\]
where \( d\omega \) is the area element on the unit \((N - 1)\)-sphere \( S \). If \( u \in H^2 B(M) \),
the integral on the left is bounded in \( r \), and the same is true, by linear
independence, of each term on the right. We conclude that \( a_{nm} = b_{nm} = c_{nm} = d_{nm} = 0 \).
The remaining terms in the expansion of \( u \) are all radial,
and by their linear independence and the boundedness of \( u \) we obtain
\( a = b = c = 0 \). Thus \( u \) is constant and \( M \in O_{H^2 B} \).

3. We proceed to show

**Theorem 3.** \( \bar{O}^N_{HP} \cap O^N_{H^2 B} \neq \emptyset \) for every \( N \).

**Proof.** First suppose \( N > 2 \). Consider the punctured \( N \)-space \( R \) with
the metric \( ds = r^{1/3}\lvert dx \rvert \), \( r = \lvert x \rvert \). The function \( \sigma(r) = r^{-(4(N-2)/3)} \) is positive
and harmonic, hence \( R \in \bar{O}_{HP} \). We now let
\[
h_{nm} = r^{p_n} S_{nm}, \quad k_{nm} = r^{q_n} S_{nm},
\]
\[
p_n, q_n = \frac{1}{2} \left[ -\frac{4}{3} (N - 2) \pm \sqrt{\frac{16}{9} (N - 2)^2 + 4n(n + N - 2)} \right],
\]
\[
u_{nm} = Ar^{p_n + 8/3} S_{nm}, \quad v_{nm} = Br^{q_n + 8/3} S_{nm},
\]
\[
\tau = \begin{cases} C \log r & \text{for } N = 4, \\ C r^{-4(N-4)/3} & \text{for } N \neq 4, \end{cases}
\]
and
\[
s = Dr^{8/3}.
\]

With this notation, the constants suitably chosen, the reasoning in §2
applies, and we have \( R \in O_{H^2B} \).

For \( N = 2 \), it is known that the disk \(|x| < 1\) can be given a conformal metric that excludes \( H^2B\)-functions (Nakai-Sario [3]), while harmonicity and hence the existence of \( HP\)-functions is not affected.

4. The Euclidean \( N\)-ball is trivially in \( \bar{O}_{HP} \cap \bar{O}_{H^2B} \) by virtue of \( h = r + 1 \in HP \) and \( r^2 \in H^2B \). We may therefore summarize our results thus far as follows:

**Theorem 4.** The totality of Riemannian \( N\)-manifolds decomposes, for every \( N \), into the disjoint nonempty classes

\[
O_{HP}^N \cap O_{H^2B}^N, \quad O_{HP}^N \cap \bar{O}_{H^2B}^N, \quad \bar{O}_{HP}^N \cap O_{H^2B}^N, \quad \bar{O}_{HP}^N \cap \bar{O}_{H^2B}^N.
\]

5. We turn to the relationship of \( O_{HP} \) to \( O_{H^2D} \).

**Theorem 5.** \( O_{HP}^N \cap O_{H^2D}^N \neq \emptyset \) for every \( N \).

**Proof.** We recall that the manifold \( M \) of §2 is in \( O_{HP} \). To see that it also is in \( O_{H^2D} \) we use again the expansion in §2 of \( u \in H^2 \), which we write as

\[
u = \sum_{n=0}^{\infty} \sum_{m=1}^{m_n} w_{nm}.
\]

Here for \( n = 0 \),

\[
w_{01} = a\tau + bs + c\sigma + d = f_{01},
\]

and for \( n > 0 \),

\[
w_{nm} = a_{nm}u_{nm} + b_{nm}v_{nm} + c_{nm}h_{nm} + d_{nm}k_{nm} = f_{nm}S_{nm},
\]

with

\[
f_{nm} = A r^{p_n} \log r + B r^{q_n}\log r + C r^{p_n} + D r^{q_n}.
\]

Choose a fixed \((n, m)\). Then for any \((k, l) \neq (n, m)\) and a fixed \( r_0 > 0 \), \( \Omega = \{x \in M|0 < r(x) < r_0\} \), the mixed Dirichlet integral over \( \Omega \) is

\[
0 = D_\Omega(h_{nm}, h_{kl}) = \text{const} \int_\Omega \text{grad } S_{nm} \cdot \text{grad } S_{kl} \, d\omega.
\]

A fortiori,

\[
D_\Omega(w_{nm}, w_{kl})
\]

\[
= \int_\Omega (\text{grad } f_{nm} \cdot \text{grad } f_{kl}) S_{nm} S_{kl} \, dV + \int_\Omega f_{nm} f_{kl} \text{grad } S_{nm} \cdot \text{grad } S_{kl} \, dV
\]

\[
= 0,
\]

and therefore
D(u) ≥ D(w_{nm}) = \text{const} \int_{0}^{\infty} f_{nm} \, dr = \infty

unless all coefficients (except perhaps \(d\)) in the expansion of \(u\) vanish.

6. We claim

**Theorem 6.** The totality of Riemannian \(N\)-manifolds decomposes, for every \(N\), into the disjoint nonempty classes

\[ O_{HP}^N \cap O_{H^2D}^N, \quad O_{HP}^N \cap \bar{O}_{H^2D}^N, \quad \bar{O}_{HP}^N \cap O_{H^2D}^N, \quad \bar{O}_{HP}^N \cap \bar{O}_{H^2D}^N. \]

**Proof.** For \(N > 2\), the reasoning in §5, with the notation of §3, gives \(R \in O_{H^2D}\), hence \(\bar{O}_{HP}^N \cap O_{H^2D}^N \neq \emptyset\). For \(N = 2\) this is known (Nakai-Sario [2]).

To see that \(O_{HP}^N \cap \bar{O}_{H^2D}^N \neq \emptyset\), consider the \(N\)-ball

\[ B_a^N = \{ |x| < 1, ds \}, \quad ds = (1 - |x|^2)^{\alpha} |dx|. \]

It was proved in Hada-Sario-Wang [1] that

\[ B_a^N \in O_G^N \iff \alpha \geq 1/(N - 2) \quad \text{for} \quad N > 2, \]

and

\[ B_a^N \in O_{H^2D}^N \iff \left\{ \begin{array}{ll}
\alpha \leq - \frac{3}{N + 2} & \text{for} \ 2 < N \leq 6, \\
\alpha \notin \left( - \frac{3}{N + 2}, \frac{5}{N - 6} \right) & \text{for} \ N > 6.
\end{array} \right. \]

In particular,

\[ B_1^N \in O_{HP}^N \cap \bar{O}_{H^2D}^N \quad \text{for} \quad 2 < N \leq 6 \]

and

\[ B_{1/(N-6)}^N \in O_{HP}^N \cap \bar{O}_{H^2D}^N \quad \text{for} \quad N > 6. \]

For \(N = 2\), the plane can be endowed with a metric which allows \(H^2D\)-functions (Nakai-Sario [2] and Sario-Wang [5]), and we have \(O_{HP}^2 \cap \bar{O}_{H^2D}^2 \neq \emptyset\).

The relation \(\bar{O}_{HP}^N \cap \bar{O}_{H^2D}^N \neq \emptyset\) is again trivial for every \(N\) in view of the Euclidean \(N\)-ball.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024