HIGHER $K$-THEORY FOR REGULAR SCHEMES

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ABSTRACT. Higher $K$-groups are defined for regular schemes, generalizing the $K$-theory of Karoubi and Villamayor. A spectral sequence is developed which shows how the $K$-groups depend on the local rings of the scheme. Applications to curves and affine surfaces are given.

Let $X$ be a regular separated scheme. If $U$ is an affine open subset of $X$, then the assignment $U \mapsto \text{BGL}(S^T(U, O_X))$ is a sheaf of Kan complexes on the Zariski site. Here $S$ denotes the suspension ring functor of Karoubi [10] and if $A$ is a ring, $A_\ast$ denotes the simplicial ring [11]

$$(A_\ast)_n = A[t_0, t_1, \ldots, t_n]/(t_0 + \cdots + t_n - 1).$$

We recall that $\pi_1 \text{BGL} A_\ast = K^{-1}A$, $i \geq 1$ [11], where the $K$-groups of Karoubi and Villamayor are indicated [10]. Also, recall that $K_0(A) \times \text{BGL}(A_\ast) \simeq \Omega \text{BGL}(SA_\ast)$ if $A$ is $K$-regular ([9], [8]). Thus there is a sheaf of Kan spectra $E(O_X)$ on $X$ associated to the pre-spectrum $U \mapsto (n \mapsto \text{BGL}(S^T(U, O_X)_n))$. Such sheaves have been studied by K. Brown [4] who has defined cohomology with coefficients in a sheaf of Kan spectra: $H^n(X, E(O_X))$, $n \in \mathbb{Z}$.

DEFINITION. $K^n(X) = H^n(X, E(O_X))$.

We remark that the spectra $E(O_X)$ are connected since $X$ is regular, so $K^i(X) = 0$ if $i > 0$. The main properties of these groups and most of the motivation for introducing them are summarized in

THEOREM 1. Let $X$ be a regular separated scheme.

1) If $U$ and $V$ are open subschemes of $X$, then there is an exact Mayer-Vietoris sequence

$$\cdots \to K^{i-1}(U \cap V) \to K^i(U \cup V) \to K^i(U) \oplus K^i(V) \to K^i(U \cap V) \to \cdots$$

2) If $X$ has finite (Krull) dimension, then there is a fourth quadrant spectral sequence of cohomological type

$$E_2^{pq} = H^p(X, K^q) \Rightarrow K^{p+q}(X).$$

Here $K^q$ is the sheaf in the Zariski site associated to the presheaf

$$U \mapsto K^q(\Gamma(U, O_X)), \quad U \text{ affine open.}$$


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(3) If $X = \text{spec } A$, then $K'(X) = K'(A)$, the Karoubi-Villamayor $K$-groups of $A$.

(4) $K'(X \times_{\text{spec } Z} \text{spec } Z[t]) = K'(X)$.

The properties (1) and (2) are formal properties of sheaves of Kan spectra [4]. Property (4) is immediate, since the Karoubi-Villamayor theory is invariant under polynomial extension. Property (3) however is a theorem whose proof depends on the main results of [7]. In addition the Krull theory of divisors enters in the description of the affine opens of $\text{spec } A$. Full details will be published elsewhere.

Properties (1) and (3) actually serve to provide an axiomatic characterization for the theory, in the category of regular separated noetherian schemes, as a simple induction argument shows. Also, since $K^n(X)$ arise as the homotopy of a spectrum, these groups are the coefficient groups (cohomology of a point) in a generalized cohomology theory of complexes.

**Corollary 2.** If $X$ is a regular curve, then there are short exact sequences

$$0 \to H^1(X, K^{-n-1}) \to K^{-n}(X) \to H^0(X, K^{-n}) \to 0, \quad all \ n \geq 0.$$

This is merely the fact that the spectral sequence of Theorem 1 degenerates at the $E_2$ level for curves.

**Proposition 3.** If $A$ is a Dedekind ring with field of fractions $F$, then the sequence

$$K_2(A) \to K_2(F) \to \coprod_{m \in \text{max } A} K_1(A/m) \to K_1(A) \to \cdots$$

is exact.

**Remark.** Exactness at $K_2F$ was shown by Bass if $A$ has only countably many maximal ideals [2]. Exactness at other points is classical.

One makes use of the recently discovered short exact sequence of K. Dennis and M. R. Stein [6] to construct a short exact sequence of sheaves

$$0 \to K^{-2} \to K_2F_X \to \coprod_v K_1(k_v) \to 0.$$

Here $K_2F_X$ is the constant sheaf where $F$ is the field of rational functions of $X$, and $\coprod_v K_1(k_v)$ assigns to each open set $U$ the group

$$\coprod_{v \in U; v \text{ closed}} K_1(k_v),$$

where $k_v$ is the residue class field at $v$. One takes the long exact cohomology sequence associated to this short exact sequence of sheaves, and splices it to the short exact sequences of Corollary 2 to get the result.
PROPOSITION 4. Let $X$ be a regular affine surface and suppose that $\xi$ is a vector bundle on $X$. Suppose in addition that $\det \xi$, the determinant bundle in $\text{Pic} \ X$, is trivial. Then $\xi$ and rank $\xi$ have the same class in $K^0(X)$ if and only if $c_2(\xi) = 0$, where $c_2(\xi) \in H^2(X, K^{-2})$ is the universal second Chern class of $\xi$.

The interpretation of the class of $\xi$ in $H^2(X, K^{-2})$ as a universal Chern class is suggested by recent work of Spencer Bloch. This result follows from the spectral sequence of Theorem 1 with the observation that the differential $d_2: H^0(X, K^{-1}) \to H^2(X, K^{-2})$ is zero, since $H^0(X, K^{-1}) = \Gamma(X, O_X^*) = U(A)$ is a direct factor of $K^{-1}(X) = K_1(A)$, where $A = \Gamma(X, O_X)$.

Denote now by $K^b_i(X), i = 0, 1$, the $K$-groups of the abelian category of coherent $O_X$ modules [1]. There are natural morphisms $K^b_i \to K^{-i}$. Of course, by Theorem 1, $K^b_i(X) = K^{-i}(X)$ if $X$ is affine and regular ($i = 0, 1$). From the Mayer-Vietoris sequence of Theorem 1 and the corresponding Mayer-Vietoris sequence for $K^b$ (which can be deduced from [5, Proposition 7]), it follows that if the regular scheme $X$ is the union of two open affines, then $K^b_0(X) = K^0(X)$. In particular this holds for curves. However, we do not know how much more generally this result holds.

Concerning $K^b_1$ the result is less satisfactory. Using the results of L. Robert's thesis [12] we can show

PROPOSITION 5. If $X$ is a complete nonsingular elliptic curve over the complex numbers, then $K^b_1(X) \to K^{-1}(X)$ is surjective but not injective.

If $X$ is a complete nonsingular curve over the constant field $k$, the algebraic closure of a finite field, then using results of Tate [13] we can show that $K^{-2}(X) = \text{Tor}(k^*, \text{Pic}(X))$ and $K^{-1}(X) = k^* \otimes \text{Pic} X \cong k^*$. The first assertion amounts to an identification of $K^{-2}(X)$ with the tame kernel in the function field case.

REFERENCES

2. —, $K_2$ des corps globaux, Séminaire Bourbaki 1971, Exposé 394.


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