BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

A CLASS OF π_c GROUPS CLOSED UNDER CYCLIC AMALGAMATIONS

BY BENNY EVANS

Communicated by J. Rotman, July 19, 1972

This note is to announce the existence of a class $\mathscr{C}$ of π_c groups (see below for definition of π_c) that is closed under the generalized free product with a single cyclic subgroup amalgamated. The class $\mathscr{C}$ has the additional property that if $A \in \mathscr{C}$ and B is any π_c group then the generalized free product $G = *(A, B; a_0 = b_0)$, where a_0 and b_0 generate isomorphic subgroups of A and B respectively, is again a π_c group. (However, it may be that $G \notin \mathscr{C}$.) In contrast we show that for each residually finite group A with an element a_0 of infinite order, there is a residually finite group B and an element b_0 in B such that the generalized free product $*(A, B; a_0 = b_0)$ is not residually finite.

The theorems above provide new proofs (as well as important generalizations) of previous theorems of P. Stebe [**2**] and G. Baumslag [**1**]. Details will appear elsewhere.

Let $\mathscr{C}$ denote the class of all π_c groups A with the property that if B is any π_c group, then the generalized free product $*(A, B; a_0 = b_0)$ is a π_c group. Recall that a group G is a π_c group (as defined in [**2**]) if, for every pair of elements g_1, g_2 of G, either $g_1 \in \langle g_2 \rangle$ or there exists a normal subgroup N of G having finite index with $\bar{g}_1 \notin \langle \bar{g}_2 \rangle v$ in G/N (bar denotes coset modulo N).

THEOREM 1. *If A and B are both in $\mathscr{C}$, and if a_0 and b_0 generate isomorphic subgroups of A and B respectively, then $*(A, B; a_0 = b_0)$ is also in the class $\mathscr{C}$.*

The proof of Theorem 1 requires a study of the finite quotient groups of π_c groups. With each element g of a group G, we associate a set $G(g)$ of positive integers with the property that $n \in G(g)$ if and only if G has a finite quotient group in which the image of g has order n. Let g be a nontrivial element of a group G. A subset X of $G(g)$ is said to be *cofinal* in $G(g)$ if for each pair $g_1, g_2 \in G$ $(g_1 \neq 1)$, either $g_1 = g_2^t$ for some t or there is a homomorphism f of G onto a finite group such that $f(g_1) \neq f(g_2)^t$ for any t, and the order of $f(g)$ is in X. In particular G is a π_c group if $G(g)$ is cofinal in $G(g)$ for some g in G. More generally, we can prove the following lemma.

AMS (MOS) subject classifications (1970). Primary 20E25.

LEMMA 1. *Let A and B be π_c groups, and let a_0 and b_0 be elements of infinite order in A and B respectively. Then the generalized free product $*(A, B; a_0 = b_0)$ is a π_c group if and only if $A(a_0) \cap B(b_0)$ is cofinal in both $A(a_0)$ and $B(b_0)$.*

Let G be a group and g an element of infinite order in G. We say that G has regular quotients at g if there is a constant K_g such that $\{nK_g | n = 1, 2, 3, \ldots\}$ is a subset of $G(g)$.

LEMMA 2. *If A is a π_c group with regular quotients at a, and if B is any π_c group, then $A(a) \cap B(b)$ is cofinal in both $A(a)$ and $B(b)$ for each b in B.*

LEMMA 3. *If $G = *(A, B; a_0 = b_0)$ is a π_c group, then G has regular quotients at each element of cyclic length greater than one in G.*

Lemmas 1, 2, and 3 may be used to obtain the main part of the proof of Theorem 1.

THEOREM 2. *Free groups, parafree groups, polycyclic groups, fundamental groups of 2-manifolds, as well as finite extensions of the above groups all belong to the class $\mathscr{C}$.*

To prove Theorem 2, we note that as a consequence of Lemmas 2 and 3 it suffices to prove that the groups in question have regular quotients at each of their elements. This may be done in most cases by examining the commutator series in a polycyclic group.

REFERENCES

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SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540