PERIODIC AND HOMOGENEOUS STATES
ON A VON NEUMANN ALGEBRA. I

BY MASAMICHI TAKESAKI

Communicated by Jack E. Feldman, June 14, 1972

This paper is devoted to announcing a structure theorem for von Neumann algebras admitting a periodic homogeneous faithful state (see Definitions 1 and 2).

Let $\mathcal{M}$ be a von Neumann algebra. Suppose that $\phi$ is a faithful normal state on $\mathcal{M}$. We denote by $\sigma_\phi^t$ the modular automorphism group of $\mathcal{M}$ associated with $\phi$. Let $G(\phi)$ denote the group of all automorphisms of $\mathcal{M}$ which leave $\phi$ invariant. We introduce the following terminologies concerning $\phi$.

**Definition 1.** If there exists $T > 0$ such that $\sigma_\phi^T$ is the identity automorphism of $\mathcal{M}$, denoted by $1$, then we call $\phi$ periodic. The smallest such number $T$ is called the period of $\phi$.

**Definition 2.** We call $\phi$ homogeneous if $G(\phi)$ acts ergodically on $\mathcal{M}$; that is, the fixed points of $G(\phi)$ are only scalar multiples of the identity.

**Definition 3.** We call $\phi$ ergodic if $\{\sigma_\phi^t\}$ is ergodic.

The ergodicity of $\phi$ implies the homogeneity of $\phi$, since $\{\sigma_\phi^t\}$ is contained in $G(\phi)$. Furthermore, if $\mathcal{M}$ admits an ergodic state, then $\mathcal{M}$ must be a factor.

Now, suppose $\phi$ is a periodic homogeneous faithful normal state on $\mathcal{M}$, which will be fixed throughout the discussion. Considering the cyclic representation of $\mathcal{M}$ induced by $\phi$, we assume that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$ with a distinguished cyclic vector $\xi_0$ such that $\phi(x) = (x\xi_0|\xi_0)$, $x \in \mathcal{M}$. According to the theory of modular Hilbert algebras (which the author proposes to call Tomita algebras), there exists the positive self-adjoint operator $\Delta$ on $\mathcal{H}$ and the unitary involution $J$ on $\mathcal{H}$ such that

$$\sigma_\phi^t(x) = \Delta^{it}x\Delta^{-it}, \quad x \in \mathcal{M};$$

$$\Delta^{it}\xi_0 = \xi_0;$$

$$J\mathcal{M}J = \mathcal{M}'; \quad J\Delta^{it}J = \Delta^{it}.$$

Put $\alpha = e^{-2\pi i/T}$ with $T$ the period of $\phi$. Obviously, we have $0 < \alpha < 1$. We introduce the following notations:

---


Key words and phrases: von Neumann algebras, modular automorphism group, periodic state, homogeneous state.

The preparation of this paper was supported in part by NSF grant GP-28737.
\[ M_n = \{ x \in M : \sigma_t^\phi(x) = \alpha^{int}x, t \in \mathbb{R} \}, \]
\[ \mathcal{H}_n = \{ \xi \in \mathcal{H} : \Delta^t \xi = \alpha^{int} \xi, t \in \mathbb{R} \}, \]

for \( n = 0, \pm 1, \pm 2, \ldots \). Then \( M_0 \) is nothing but the centralizer \( M_{\phi} \) of \( \phi \) in the sense of [11, Definition 8.6]. The ergodicity of \( G(\phi) \) implies that \( M_n \neq \{0\} \) for every integer \( n \). The subspace \( M_n \) of \( M \) is also given by

\[ \mathcal{H}_n = \{ x \in \mathcal{H} : \phi(xy) = \alpha^n \phi(yx) \text{ for every } y \in \mathcal{M} \}, \]

due to Størmer [9].

**Lemma 4.** We have the following:

(i) \( M_n M_m \subseteq M_{n+m} \), \( M_n^* = M_{-n} \);

(ii) \( M_n \mathcal{H}_m \subseteq \mathcal{H}_{n+m} \), \( J \mathcal{H}_n = \mathcal{H}_{-n} \);

(iii) \( \mathcal{H} = \sum_{n=-\infty}^{\infty} \mathcal{H}_n \);

(iv) \( \mathcal{H}_n = [\mathcal{H} \cap \mathcal{H}_0]^n \).

It is easily seen that the algebraic direct sum \( \sum_{n=-\infty}^{\infty} M_n \) is a \( \sigma \)-weakly dense *-subalgebra of \( M \). If \( N \) is a von Neumann subalgebra of \( M \) invariant under \( \sigma_t^\phi \), then the algebraic direct sum \( \sum_{n=-\infty}^{\infty} (N \cap M_n) \) is also a \( \sigma \)-weakly dense *-subalgebra of \( N \). Since \( M_n^* M_n \subseteq M_0 \) and \( M_n M_n^* \subseteq M_0 \), the absolute value \( |x| \) of every element \( x \) in \( M_n \) falls in \( M_0 \). Hence, if \( x \in M_n \) commutes with \( M_0 \), then \( x \) commutes with \( x^*x \) and \( xx^* \), so that \( x \) is normal, that is, \( x^*x = xx^* \). But this is impossible unless \( x \) is in \( M_0 \) because \( \alpha^n \phi(x^*x) = \phi(xx^*) \). Thus we obtain the following:

**Proposition 5.** The relative commutant \( M_0' \cap M \) of \( M_0 \) in \( M \) is contained in \( M_0 \) as the center of \( M_0 \), denoted by \( \mathcal{Z}_0 \).

We denote by \( \pi_n \) the normal representation of \( M_0 \) on \( \mathcal{H}_n \) defined by restricting the action of \( M_0 \) to \( \mathcal{H}_n \). We also define the antirepresentation \( \pi_n' \) of \( M_0 \) on \( \mathcal{H}_n \) by

\[ \pi_n(a) = J\pi_{-n}(a)^*J, \quad a \in M_0. \]

For each \( x \in M_n \), we have

\[ \pi_n(a)x \xi_0 = ax \xi_0; \]
\[ \pi_n'(a)x \xi_0 = xa \xi_0, \quad x \in M_0. \]

Hence \( \pi_n \) and \( \pi_n' \) commute. Making use of the ergodicity of \( G(\phi) \), we can prove the following:

**Lemma 6.** Both \( \pi_n \) and \( \pi_n' \) are faithful.

For each \( g \in G(\phi) \), we define a unitary operator \( U(g) \) on \( \mathcal{H} \) by

\[ U(g)x \xi_0 = g(x) \xi_0, \quad x \in M. \]

Then the map \( :g \in G(\phi) \mapsto U(g) \) is a representation of \( G(\phi) \) and covariant.
with the action of \( \mathcal{M} \). It is easily seen that
\[
U(g)\pi_n(x)U(g)^* = \pi_n \circ g(x); \\
U(g)\pi_n(x)U(g)^* = \pi_n \circ g(x), \quad x \in \mathcal{M}_0, g \in G(\phi).
\]
The ergodicity of \( G(\phi) \) on \( \mathcal{M}_0 \) yields that the coupling operator of \( \{\pi_n(\mathcal{M}_0), \mathcal{S}_n\} \) in the sense of Griffin [6] is a scalar multiple of the identity. Therefore, \( \{\pi_n(\mathcal{M}_0), \mathcal{S}_n\} \) has either a separating vector or a cyclic vector.

**Lemma 7.** For \( n \geq 1 \), \( \{\pi_n, \mathcal{S}_n\} \) does not have a separating vector.

**Proof.** Since every \( \xi \in \mathcal{S}_n \) is analytic for \( \Delta^u \), there exists a closed operator \( a \) affiliated with \( \mathcal{M} \) such that \( \xi = a\zeta_0 \). We can choose \( a \) so that \( \Delta^u a \Delta^{-u} = x^m a \). Let \( h = uh \) be the polar decomposition of \( a \). Then \( h \) is affiliated with \( \mathcal{M}_0 \) and \( u \in \mathcal{M}_n \). If \( \zeta \) is separating, then \( x\zeta = 0, x \in \mathcal{M}_0 \), implies \( x = 0 \), so that \( xu \in 0 \) implies \( x = 0 \). Hence \( uu^* = 1 \). But \( x^\phi(u^*u) = \phi(u^*u) = 1 \), so that \( \phi(u^*u) = x^{-n} > 1 \) if \( n \geq 1 \), a contradiction.

Therefore, \( \{\pi_n, \mathcal{S}_n\}, n \geq 1 \), has a cyclic vector \( \xi \), which is separating for \( \pi_{-n}(\mathcal{M}_0) \). If \( a = ku \) is the right polar decomposition of the above \( a \) in Lemma 7, then \( ux = 0, x \in \mathcal{M}_0 \), implies \( x = 0 \), so that we have \( u^*u = 1 \), and \( \phi(u^*u) = x^n \). We choose an element \( u_1 \) in \( \mathcal{M}_1 \) with \( u_1^*u_1 = 1 \), and fix it. Then \( u_1^* \) falls in \( \mathcal{M}_n \) for \( n \geq 1 \), and \( \mathcal{M}_n = \mathcal{M}_0u_1^* \) because \( \mathcal{M}_n u_1^* \in \mathcal{M}_0 \). Therefore we have
\[
\mathcal{M}_n = \mathcal{M}_0u_1^*; \\
\mathcal{M}_{-n} = u_1^*n\mathcal{M}_0, \quad n = 1, 2, \ldots.
\]
Thus the von Neumann algebra \( \mathcal{M} \) is generated by \( \mathcal{M}_0 \) and the isometry \( u_1 \). The choice of \( u_1 \) is unique in the following sense:

**Lemma 8.** Every partial isometry \( v \) in \( \mathcal{M}_1 \) is of the form \( wu_1 \) with a partial isometry \( w \) in \( \mathcal{M}_0 \).

Let \( e_{-n} \) denote the projections \( u_1^*u_1^* \) in \( \mathcal{M}_0 \) for \( n \geq 1 \). Then Lemma 8 implies, together with the ergodicity of \( G(\phi) \), that
\[
e_{-n} = x^n 1.
\]
Thus we conclude that \( \mathcal{M}_0 \) is of type \( II_1 \). We denote by \( e_n \) the projection \( Je_{-n}J \) in \( \mathcal{M}_0 \). Let \( \mathcal{S}_0 = e_n \mathcal{S}_0 \), for every integer \( n \).

Define an isomorphism \( \theta \) of \( \mathcal{M}_0 \) onto \( e_{-1}\mathcal{M}_0e_{-1} \) by \( \theta(x) = u_1xu_1^* \), \( x \in \mathcal{M}_0 \). Then the isomorphism \( \theta \) induces an automorphism \( \tilde{\theta} \) of \( \mathcal{L}_0 \) by the equality \( \theta(a) = \tilde{\theta}(a)e_{-1}, a \in \mathcal{L}_0 \). It follows from Lemma 8 that \( \tilde{\theta} \) does not depend on the choice of \( u_1 \).

**Proposition 9.** The center \( \mathcal{L} \) of \( \mathcal{M} \) is precisely the fixed point subalgebra of \( \mathcal{L}_0 \) with respect to \( \tilde{\theta} \). Therefore, \( \mathcal{M} \) is a factor if and only if \( \tilde{\theta} \) is ergodic on \( \mathcal{L}_0 \).
PROPOSITION 10. For $n \geq 1$, we have
\[ \{\pi_n, S_n\} \cong \{\pi_0, R_n\}; \]
\[ \{\pi_{-n}, S_{-n}\} \cong \{\theta^n, R_{-n}\}, \]
where $\{\pi_0, R_n\}$ means the restriction of $\pi_0$ to the invariant subspace $R_n$.

We denote by $\phi_0$ the restriction of $\phi$ to $\mathcal{M}$.

THEOREM 11. In the pre-Hilbert space metric given by the state $\phi$, the von Neumann algebra $\mathcal{M}$ is decomposed as
\[ \mathcal{M} = \cdots \oplus u_{1}^n \mathcal{M}_0 \oplus \cdots \oplus u_{1}^{*} \mathcal{M}_0 \oplus \mathcal{M}_0 u_{1} \oplus \cdots \oplus \mathcal{M}_0 u_{1}^{*} \oplus \cdots. \]

The algebraic structure of $(\mathcal{M}, \phi)$ is determined by $\{\mathcal{M}_0, \theta, \phi_0\}$ in the following sense: Let $\tilde{\mathcal{M}}$ be another von Neumann algebra equipped with a periodic homogeneous faithful state $\tilde{\phi}$ of period $T$ and let $\tilde{\mathcal{M}}$ be decomposed with respect to $\tilde{\phi}$ as
\[ \tilde{\mathcal{M}} = \cdots \oplus \tilde{u}_1 \tilde{\mathcal{M}}_0 \oplus \cdots \oplus \tilde{u}_1^{*} \tilde{\mathcal{M}}_0 \oplus \tilde{\mathcal{M}}_0 \tilde{u}_1 \oplus \cdots \oplus \tilde{\mathcal{M}}_0 \tilde{u}_1^{*} \oplus \cdots. \]

Suppose $\tilde{u}_1$ gives rise to an isomorphism of $\theta$ of $\tilde{\mathcal{M}}_0$ onto $\tilde{\mathcal{M}}_0 \tilde{\mathcal{M}}_0 \tilde{\mathcal{M}}_0^{-1}$. Then there exists an isomorphism $\sigma$ of $\mathcal{M}$ onto $\tilde{\mathcal{M}}$ with $\phi = \phi \circ \sigma$ if and only if there exists an isomorphism $\sigma_0$ of $\mathcal{M}_0$ onto $\tilde{\mathcal{M}}_0$ and a partial isometry $w$ in $\mathcal{M}_0$ such that $w \theta(x) w^* = \sigma_0^{-1} \circ \theta \circ \sigma_0(x)$, $x \in \mathcal{M}_0$, and $\phi_0 = \phi_0 \circ \sigma$, where $\phi_0$ (resp. $\phi_0$) means the restriction of $\phi$ (resp. $\phi$) to $\mathcal{M}_0$ (resp. $\tilde{\mathcal{M}}_0$).

Conversely, if $\mathcal{M}_0$ is a von Neumann algebra of type $II_1$. Let $e$ be a projection of $\mathcal{M}_0$ with $e^2 = \alpha$, $0 < \alpha < 1$. Suppose $\theta$ is an isomorphism of $\mathcal{M}_0$ onto $e \mathcal{M}_0 e$. Then $\theta$ induces an automorphism $\tilde{\theta}$ of the center $\mathcal{Z}$ of $\mathcal{M}_0$ such that $\tilde{\theta}(a)e = \theta(a)e$, $a \in \mathcal{Z}_0$. Let $\phi_0$ be a $\tilde{\theta}$-invariant faithful normal state on $\mathcal{Z}_0$. We extend $\phi_0$ to a faithful normal trace on $\mathcal{M}_0$ by $\phi_0(x) = \phi_0(x^e)$, $x \in \mathcal{M}_0$. Suppose $G$ denotes the group of all automorphisms $g$ of $\mathcal{M}_0$ such that there exists a partial isometry $w_g$ in $\mathcal{M}_0$ with $g \circ \theta \circ g^{-1}(x) = w_g \theta(x) w_g^*$, and such that $\phi_0 \circ g = \phi_0$ (this is satisfied automatically if $\tilde{\theta}$ is ergodic). Such an automorphism is called admissible.

THEOREM 12. In the above situation, if $G$ acts ergodically on the center $\mathcal{Z}_0$, then there exists a von Neumann algebra $\mathcal{M}$ with a periodic homogeneous faithful state $\phi$ of period $T = -2\pi/\log \alpha$ such that $\{\mathcal{M}_0, \theta, \phi_0\}$ appears in the decomposition of $\mathcal{M}$ associated with $\phi$ as described in Theorem 11.

We denote by $\mathcal{R}(\mathcal{M}_0, \theta, \phi_0)$ the von Neumann algebra determined by $(\mathcal{M}_0, \theta, \phi_0)$ in Theorems 11 and 12. We can describe the automorphism group $G(\phi)$ in terms of $G$ and the unitary group of $\mathcal{Z}_0$. In order to distinguish the algebraic type of $\mathcal{R}(\mathcal{M}_0, \theta, \phi_0)$, we employ new results of A. Connes [4] concerning modular automorphism groups.
For a von Neumann algebra \( M \), let \( \text{Aut}(M) \) (resp. \( \text{Int}(M) \)) denote the group of all (resp. inner) automorphisms of \( M \). Let \( \text{Out}(M) \) denote the quotient group \( \text{Aut}(M)/\text{Int}(M) \). A. Connes showed recently that the canonical image \( \sigma^\phi_t \) of the modular automorphism group \( \sigma^\phi_t \) in \( \text{Out}(M) \) does not depend on the choice of \( \phi \); hence we denote it simply by \( \sigma_t \). Furthermore he proved that if \( \sigma^\phi_t \) is inner for some \( T > 0 \), then \( \sigma_t \) is given by a unitary operator in the center of the centralizer \( M_\phi \) of \( \phi \).

Now, we return to the original situation. In order to avoid any possible confusion, we denote by \( T_0 \) the period of our state \( \phi \).

**Theorem 13.** For \( T > 0 \), \( \sigma_t \) is inner, that is, \( \sigma_T = \text{identity} \), if and only if \( \alpha^{-IT} \) is a point spectrum of the automorphism \( \tilde{\theta} \) of \( \mathcal{Z}_0 \).

Therefore, if we have ergodic automorphisms \( \tilde{\theta} \) in \( \mathcal{Z}_0 \) of different point spectral type, then the resulting factors \( \mathcal{Z}(\mathcal{M}_0, \theta, \phi_0) \) are nonisomorphic.

**Examples.** Let \( \mathcal{F} \) denote a hyperfinite II\(_1\) factor and \( \mathcal{A} = L^\infty(0,1) \). Let \( \mathcal{M}_0 = \mathcal{F} \otimes \mathcal{A} \). For \( 0 < \alpha < 1 \), we choose a projection \( f \in \mathcal{F} \) with \( \tau(f) = \alpha \), where \( \tau \) is the canonical trace of \( \mathcal{F} \). It is then known that there exists an isomorphism \( \theta_1 \) of \( \mathcal{F} \) onto \( f \mathcal{F} f \). Let \( \tilde{\theta} \) be an ergodic automorphism of \( \mathcal{A} \) with invariant faithful normal state \( \mu \). Let \( \theta_0 = \theta_1 \otimes \tilde{\theta} \) and \( \phi_0 = \tau \otimes \mu \). Then the triplet \( \{ \mathcal{M}_0, \theta, \phi_0 \} \) satisfies all our requirements, since the automorphism \( \text{id} \otimes \tilde{\theta}^n, n = 0 \pm 1, \pm 2, \ldots \), are admissible and ergodic on the center \( \mathcal{Z}_0 = 1 \otimes \mathcal{A} \). Thus, if we choose various kinds of ergodic automorphisms \( \tilde{\theta} \), then we get different kinds of modular groups \( \sigma_t \) as well as different factors.

**References**


**Department of Mathematics, University of California, Los Angeles, California 90024**