ON OPERATOR-VALUED STOCHASTIC INTEGRALS

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1. Introduction. The purpose of this note is to announce some theorems concerning the operator-valued stochastic integrals which arise naturally from the study of the regularity properties of solutions of stochastic integral equations. Proofs and detailed discussion will be given in [7]. Let $B^* \subset H \subset B$ be an abstract Wiener space [1] and $W(t)$ a Wiener process in $B$. Various stochastic integrals associated with $W(t)$ were studied and an infinite-dimensional version of Ito's formula [2] was proved in [5]. This formula was used to connect the solution of a stochastic integral equation with the corresponding heat equation [8]. In [6] we proved another version of Ito's formula which was used to construct diffusion processes, in particular, a Brownian motion, in a Riemann-Wiener manifold. We present here a third version of Ito's formula and use it to prove an infinite-dimensional analogue of a formula on p. 58 of McKean's book [9]. An operator-valued stochastic integral has been studied by Kannan and Bharucha-Reid [4]. However, there appears to be no relation between their work and ours in this paper.

2. Notation and definitions. Let $X$ and $Y$ be two real Banach spaces. $L^n(X; Y)$ denotes the Banach space of all continuous $n$-linear maps from $X^n$ into $Y$ with the usual norm $\| \cdot \|_{X^n; Y}$. $L^1$ will be written as $L^1(H; X^*)$. $L^2(X; R)$ will be identified as $L^2(H; R)$ in a well-known way. $L^2(H; R)$ denotes the Hilbert space of all Hilbert-Schmidt operators of $H$ with H-S-norm $\| \cdot \|_2 = \langle \langle \cdot , \cdot \rangle \rangle^{1/2}$. We have the relation $L^2(B; R) = L^2(H; R)$.

DEFINITION. Let $K$ be a Hilbert space with inner product $(\cdot , \cdot )$. A continuous bilinear map $S$ from $H \times H$ into $K$ is said to be of trace class type if (i) for each $x \in K$, $S_x$ is a trace class operator of $H$, where $S_x(\cdot , \cdot ) = (S(\cdot , \cdot ), x)$ and (ii) the functional $x \rightarrow \text{trace } S_x$ is continuous. $\mathcal{S}(H; K)$ denotes the space consisting of all such continuous bilinear maps.

Let $S$ be of trace class type. Then there is a unique element, denoted by $\text{TRACE } S$, of $K$ such that $(\text{TRACE } S, x) = \text{trace } S_x$ for all $x \in K$. Note that $L^2(B; L(B, B^*)) \subset \mathcal{S}(H; L^2(H; H))$. 

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DEFINITION. Let $T \in L^2(H; R)$. Define $\hat{T} \in L(H; L(H, H))$ by $\hat{T}(h) = T(h, \cdot, \cdot)$. $T$ is said to be of Hilbert-Schmidt type if (i) $\hat{T}(H) \subset L^2(H, H)$ and (ii) $\hat{T}$ is a Hilbert-Schmidt operator from $H$ into $L^2(H, H)$. $L^2(H; R)$ denotes the space consisting of all such $T$'s.

Define $\|T\|_2$ to be the H-S-norm of $T$. Clearly, $\|T\|_2^2 = \sum_{i,j,k} T(e_i, e_j, e_k)^2$ for any O.N.B. $\{e_k\}_{k=1}^\infty$ of $H$. $L^2(H; R)$ is a Hilbert space. Moreover, we have the relation $L^2(B; R) \subset \mathcal{F}(H; H) \subset L^2(H; R)$.

3. Theorems. As in [5] we assume the following on the abstract Wiener space $B^* \subset H \subset B$: There exists a sequence $Q_n$ of finite-dimensional projections such that (i) $Q_n(B) \subset B^*$ and (ii) $Q_n$ converges strongly to the identity both in $B$ and in $H$.

$L(H, H)$ has three topologies, namely, the uniform topology, strong topology and weak topology. However, the Borel fields corresponding to them are all the same. This can be shown by a similar argument used in [3]. Therefore, when we talk about the measurability of a random variable (resp. a process) with values in (resp. state space) $L(H, H)$ there is no need to specify the Borel field. We have the same situation for $L^3(B; R) \approx L^3(B; R)$. Let $\xi(t)$ be a simple nonanticipating (with respect to the Wiener process $W(t)$) process with state space $L^3(B; R)$. Suppose $\xi$ has jumps at $0 < t_1 < \cdots < t_n$. Define

$$I_\xi(t) = \sum_{k=0}^{n-1} \xi(t_k)(W(t_{k+1}) - W(t_k)) + \xi(t_j)(W(t) - W(t_j)),$$

if $t_j \leq t < t_{j+1}$. Here $t_0 = 0$ and $t_{n+1} = \infty$. We will regard $I_\xi(t)$ as a process with state space $L(H, H)$ via the inclusions $L(B; B^*) \subset L^3(B; R) \subset L(H, H)$.

**Theorem 1.** For every nonanticipating process $\xi(t)$ with state space $L^3(B; R)$ such that

$$\int_0^\tau \|\xi(t)\|_{L^3(B)}^2 dt < \infty \quad \text{a.s. for each } 0 < \tau < \infty,$$

we can determine a stochastic process $I_\xi(t) = \int_0^t \xi(s) dW(s)$ satisfying the following properties:

(i) $I_\xi$ has continuous sample paths (uniform topology for $L(H, H)$);

(ii) $I_\xi$ is a martingale;

(iii) $\text{prob}\{\sup_{0 \leq t \leq \delta} \|I_\xi(t)\|_2 > \delta\} \leq \delta^{-2} E\{\|I_\xi(\tau)\|_2^2\}$

(iv) $E\{I_\xi(t)\} = 0$ and $E\{\|I_\xi(t)\|_2^2\} = E\{\int_0^\tau \|\xi(s)\|_2^2 ds\}$

(v) $I_{\alpha_1} + \beta I_{\xi_2} = \alpha I_{\xi_1} + \beta I_{\xi_2}$

(vi) $I_\xi$ is nonanticipating.

Let $S \in L(L(H; H); L(H; H))$ and $T \in L^3(H; R)$. Define $S \triangle T \in L^3(H; R)$ by $(S \triangle T) = S \circ \hat{T}$. Note that if $L(B; B^*)$ is invariant under $S$ and $T \in L^3(B; R)$ then $S \triangle T \in L^3(B; R)$. Note also that if $T \in L^3(B; R)$ and $U \in L^2(L(H; H); L(H; H))$ is such that $U(L(B; B^*) \times L(B; B^*)) \subset$...
L(B; B*) then \( U \circ [T \times T] \in L^2(B; L(B; B*)) \subset \mathcal{S}(H; L_{(2)}(H; H)) \) so that we have \( \text{TRACE} \ U \circ [T \times T] \in L_{(2)}(H; H) \).

**Theorem 2 (Itô's Formula).** Let \( \theta \) be a \( C^2 \)-map from \( L(H; H) \) into itself such that \( \theta'(S)(L(B; B*)) \subset L(B; B*) \) and \( \theta''(S)(L(B; B*) \times L(B; B*)) \subset L(B; B*) \) for all \( S \in L(H; H) \). If \( dX(t) = \xi(t) \, dW(t) + \zeta(t) \, dt \), where \( \xi(t) \) is a nonanticipating process in \( L^2(B; R) \) such that \( \int_0^t \| \xi(t) \|^2_{B; B} \, dt < \infty \) a.s. for each \( 0 < \tau < \infty \) and \( \zeta(t) \) is a nonanticipating process in \( L(H; H) \) with \( \int_0^t \| \zeta(t) \|^2_{H; H} \, dt < \infty \) a.s. for each \( 0 < \tau < \infty \) then

\[
d\theta(X(t)) = \theta'(X(t)) \triangledown \xi(t) \, dW(t) + \theta''(X(t)) \circ \left\{ \frac{1}{2} \text{TRACE} \, \theta''(X(t)) \circ \{ \xi(t) \times \xi(t) \} \right\} \, dt.
\]

**Theorem 3.** Let \( f \) and \( g \) be maps from \( [t_0, \infty) \times L(H; H) \) into \( L^3(B; R) \) and \( L(H; H) \), respectively \( (t_0 \geq 0) \). Assume that \( f \) and \( g \) satisfy the following conditions:

(i) for each \( S \in L(H; H) \), \( f(\cdot, S, \cdot) \) and \( g(\cdot, S, \cdot) \) are nonanticipating;

(ii) there is a constant \( c \) such that, with probability 1,

\[
\| f(t, S) - f(t, T) \|_2 + \| g(t, S) - g(t, T) \|_{H; H} \leq c \| S - T \|_{H; H},
\]

and

\[
\| f(t, S) \|_2^2 + \| g(t, S) \|_{H; H}^2 \leq c(1 + \| S \|_{H; H}^2)
\]

for all \( t \in [t_0, \infty) \) and \( S, T \in L(H; H) \).

Then the \( L(H; H) \)-valued stochastic differential equation

\[
dY(t) = f(t, Y(t)) \, dW(t) + g(t, Y(t)) \, dt
\]

has a unique nonanticipating continuous solution. The solution is a Markov process.

**4. An application.** If \( S \in L(H; H) \) and \( T \in L^3(H; R) \) then we define \( S \triangle T \in L^3(H; R) \) by \( (S \triangle T)(x) = S \circ (T(x)), x \in H \). Consider the stochastic integral equation

\[
X(t) = I + \int_0^t X(s) \triangle \xi(s) \, dW(s) + \int_0^t X(s) \circ \eta(s) \, ds,
\]

where \( \xi \) and \( \eta \) are bounded nonanticipating processes with state spaces \( L^3(B; R) \) and \( L(H; H) \), respectively.

**Theorem 4 (Girsanov-Skorokhod-McKean’s Formula).** Suppose that, with probability 1, \( \{ \xi(t)(x), \eta(t); 0 \leq t < \infty, x \in B \} \) forms a commutative family of operators. Then the solution of the above equation can be represented by

\[
X(t) = \exp \left\{ \int_0^t \xi(s) \, dW(s) + \int_0^t \{ \eta(s) - \frac{1}{2} \text{TRACE} \, (\kappa \circ [\xi(s) \times \xi(s)]) \} \, ds \right\},
\]

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where $\kappa$ is the map from $L(H; H) \times L(H; H)$ into $L(H; H)$ given by $\kappa(S, T) = S \circ T$.

REFERENCES


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