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ON OPERATOR-VALUED STOCHASTIC INTEGRALS

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1. **Introduction.** The purpose of this note is to announce some theorems concerning the operator-valued stochastic integrals which arise naturally from the study of the regularity properties of solutions of stochastic integral equations. Proofs and detailed discussion will be given in [**7**]. Let $B^* \subset H \subset B$ be an abstract Wiener space [**1**] and $W(t)$ a Wiener process in B. Various stochastic integrals associated with $W(t)$ were studied and an infinite-dimensional version of Ito's formula [**2**] was proved in [**5**]. This formula was used to connect the solution of a stochastic integral equation with the corresponding heat equation [**8**]. In [**6**] we proved another version of Ito's formula which was used to construct diffusion processes, in particular, a Brownian motion, in a Riemann-Wiener manifold. We present here a third version of Ito's formula and use it to prove an infinite-dimensional analogue of a formula on p. 58 of McKean's book [**9**]. An operator-valued stochastic integral has been studied by Kannan and Bharucha-Reid [**4**]. However, there appears to be no relation between their work and ours in this paper.

2. **Notation and definitions.** Let X and Y be two real Banach spaces. $L^n(X; Y)$ denotes the Banach space of all continuous n-linear maps from X^n into Y with the usual norm $\|\cdot\|_{X^n;Y}$. L^1 will be written as L. $L^{n-1}(X; X^*)$ will be identified as $L^n(X; R)$ in a well-known way. $L^2_{(2)}(H; R)$ $(\equiv L_{(2)}(H; H))$ denotes the Hilbert space of all Hilbert-Schmidt operators of H with H-S-norm $\|\cdot\|_2 = \langle\langle\, ,\rangle\rangle^{1/2}$. We have the relation $L^2(B; R) \subset L^2_{(2)}(H; R)$.

DEFINITION. Let K be a Hilbert space with inner product $(\, ,)$. A continuous bilinear map S from $H \times H$ into K is said to be of *trace class type* if (i) for each $x \in K$, S_x is a trace class operator of H, where $S_x(\cdot,\cdot) = (S(\cdot,\cdot),x)$ and (ii) the functional $x \to \operatorname{trace} S_x$ is continuous. $\mathscr{S}(H; K)$ denotes the space consisting of all such continuous bilinear maps.

Let S be of trace class type. Then there is a unique element, denoted by TRACE S, of K such that (TRACE S, x) = trace S_x for all $x \in K$. Note that $L^2(B; L(B, B^*)) \subset \mathscr{S}(H; L_{(2)}(H; H))$.

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DEFINITION. Let $T \in L^3(H;R)$. Define $\tilde{T} \in L(H;L(H,H))$ by $\tilde{T}(h) = T(h,\cdot,\cdot)$. T is said to be of *Hilbert-Schmidt type* if (i) $\tilde{T}(H) \subset L_{(2)}(H,H)$ and (ii) $\tilde{T}$ is a Hilbert-Schmidt operator from H into $L_{(2)}(H,H)$. $L^3_{(2)}(H;R)$ denotes the space consisting of all such T's.

Define $|||T|||_2$ to be the H-S-norm of $\tilde{T}$. Clearly, $|||T|||_2^2 = \sum_{i,j,k} T(e_i, e_j, e_k)^2$ for any O.N.B. $\{e_k\}_{k=1}^{\infty}$ of H. $L^3_{(2)}(H;R)$ is a Hilbert space. Moreover, we have the relation $L^3(B;R) \subset \mathscr{S}(H;H) \subset L^3_{(2)}(H;R)$.

3. **Theorems.** As in [**5**] we assume the following on the abstract Wiener space $B^* \subset H \subset B$: There exists a sequence Q_n of finite-dimensional projections such that (i) $Q_n(B) \subset B^*$ and (ii) Q_n converges strongly to the identity both in B and in H.

$L(H,H)$ has three topologies, namely, the uniform topology, strong topology and weak topology. However, the Borel fields corresponding to them are all the same. This can be shown by a similar argument used in [**3**]. Therefore, when we talk about the measurability of a random variable (resp. a process) with values in (resp. state space) $L(H,H)$ there is no need to specify the Borel field. We have the same situation for $L(B;L(B,B^*)) \approx L^3(B;R)$. Let $\xi(t)$ be a simple nonanticipating (with respect to the Wiener process $W(t)$) process with state space $L^3(B;R)$. Suppose ξ has jumps at $0 < t_1 < \cdots < t_n$. Define

$$I_\xi(t) = \textstyle\sum_{k=0}^{j-1} \tilde{\xi}(t_k)(W(t_{k+1}) - W(t_k)) + \tilde{\xi}(t_j)(W(t) - W(t_j)),$$

if $t_j \leqq t < t_{j+1}$. Here $t_0 = 0$ and $t_{n+1} = \infty$. We will regard $I_\xi(t)$ as a process with state space $L(H,H)$ via the inclusions $L(B;B^*) \subset L_{(2)}(H,H) \subset L(H,H)$.

THEOREM 1. *For every nonanticipating process $\xi(t)$ with state space $L^3(B;R)$ such that $\int_0^\tau \|\xi(t)\|^2_{B^3;R}\,dt < \infty$ a.s. for each $0 < \tau < \infty$, we can determine a stochastic process*

$$I_\xi(t) \equiv \int_0^t \xi(s)\,dW(s)$$

satisfying the following properties:

(i) *I_ξ has continuous sample paths* (*uniform topology for $L(H,H)$*);

(ii) *I_ξ is a martingale*;

(iii) $\text{prob}\{\sup_{0 \leqq t \leqq \tau} \|I_\xi(t)\|_2 > \delta\} \leqq \delta^{-2} E\{\|I_\xi(\tau)\|_2^2\}$;

(iv) $E\{I_\xi(t)\} = 0$ *and* $E\{\|I_\xi(t)\|_2^2\} = E\int_0^t |||\xi(s)|||_2^2\,ds$;

(v) $I_{\alpha\xi_1+\beta\xi_2} = \alpha I_{\xi_1} + \beta I_{\xi_2}$;

(vi) *I_ξ is nonanticipating.*

Let $S \in L(L(H;H);L(H;H))$ and $T \in L^3(H;R)$. Define $S \triangle T \in L^3(H;R)$ by $(S \triangle T)^\sim = S \circ \tilde{T}$. Note that if $L(B;B^*)$ is invariant under S and $T \in L^3(B;R)$ then $S \triangle T \in L^3(B;R)$. Note also that if $T \in L^3(B;R)$ and $U \in L^2(L(H;H);L(H;H))$ is such that $U(L(B;B^*) \times L(B;B^*)) \subset$

$L(B; B^*)$ then $U \circ [T \times T] \in L^2(B; L(B; B^*)) \subset \mathscr{S}(H; L_{(2)}(H; H))$ so that we have TRACE $U \circ [T \times T] \in L_{(2)}(H; H)$.

THEOREM 2 (ITO'S FORMULA). *Let θ be a C^2-map from $L(H; H)$ into itself such that $\theta'(S)(L(B; B^*)) \subset L(B; B^*)$ and $\theta''(S)(L(B; B^*) \times L(B; B^*)) \subset L(B; B^*)$ for all $S \in L(H; H)$. If $dX(t) = \xi(t)\, dW(t) + \zeta(t)\, dt$, where $\xi(t)$ is a nonanticipating process in $L^3(B; R)$ such that $\int_0^\tau \|\xi(t)\|_{B;R}^2\, dt < \infty$ a.s. for each $0 < \tau < \infty$ and $\zeta(t)$ is a nonanticipating process in $L(H; H)$ with $\int_0^\tau \|\zeta(t)\|_{H,H}\, dt < \infty$ a.s. for each $0 < \tau < \infty$ then*

$$d\theta(X(t)) = \theta'(X(t)) \triangle \xi(t) dW(t) + \{\theta'(X(t))(\zeta(t)) + \tfrac{1}{2}\mathrm{TRACE}\, \theta''(X(t)) \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)]\}\, dt.$$

THEOREM 3. *Let f and g be maps from $[t_0, \infty) \times L(H; H) \times \Omega$ into $L^3(B; R)$ and $L(H; H)$, respectively ($t_0 \geqq 0$). Assume that f and g satisfy the following conditions:*

(i) *for each $S \in L(H; H)$, $f(\cdot, S, \cdot)$ and $g(\cdot, S, \cdot)$ are nonanticipating;*

(ii) *there is a constant c such that, with probability* 1,

$$|||f(t, S) - f(t, T)|||_2 + \|g(t, S) - g(t, T)\|_{H;H} \leqq c\|S - T\|_{H;H},$$

and

$$|||f(t, S)|||_2^2 + \|g(t, S)\|_{H;H}^2 \leqq c(1 + \|S\|_{H;H}^2)$$

for all $t \in [t_0, \infty)$ and $S, T \in L(H; H)$.

Then the $L(H; H)$-valued stochastic differential equation

$$dY(t) = f(t, Y(t))\, dW(t) + g(t, Y(t))\, dt$$

has a unique nonanticipating continuous solution. The solution is a Markov process.

4. **An application.** If $S \in L(H; H)$ and $T \in L^3(H; R)$ then we define $S \triangle T \in L^3(H; R)$ by $(S \triangle T)^{\sim}(x) = S \circ (\tilde{T}(x))$, $x \in H$. Consider the stochastic integral equation

$$X(t) = I + \int_0^t X(s) \triangle \xi(s)\, dW(s) + \int_0^t X(s) \circ \eta(s)\, ds,$$

where ξ and η are bounded nonanticipating processes with state spaces $L^3(B; R)$ and $L(H; H)$, respectively.

THEOREM 4 (GIRSANOV-SKOROKHOD-MCKEAN'S FORMULA). *Suppose that, with probability 1, $\{\tilde{\xi}(t)(x), \eta(t); 0 \leqq t < \infty, x \in B\}$ forms a commutative family of operators. Then the solution of the above equation can be represented by*

$$X(t) = \exp\left\{\int_0^t \xi(s)\, dW(s) + \int_0^t \{\eta(s) - \tfrac{1}{2}\mathrm{TRACE}\, (\kappa \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)])\}\, ds\right\},$$

where κ is the map from $L(H; H) \times L(H; H)$ into $L(H; H)$ given by $\kappa(S, T) = S \circ T$.

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