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# FINITELY GENERATED SUBMODULES OF DIFFERENTIABLE FUNCTIONS. II

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1. **Introduction.** Let $\mathscr{E}(\Omega)$ denote the space of real-valued infinitely differentiable functions on an open set $\Omega$ in $\mathscr{R}^n$ equipped with the topology of uniform convergence of all derivatives on all compact subsets of $\Omega$. Throughout we assume that $\Omega$ is connected.

Let $[\mathscr{E}(\Omega)]^p$ denote the Cartesian product of $\mathscr{E}(\Omega)$ with itself $p$-times equipped with the product topology. Then $[\mathscr{E}(\Omega)]^p$ is a Frechet space and a $\mathscr{E}(\Omega)$-module. In [3], the finitely generated submodules of $[\mathscr{E}^m(\Omega)]^p$ which are closed in $[\mathscr{E}^m(\Omega)]^p$ are characterized for $m < \infty$ and we are here concerned with the same problem for $m = \infty$.

2. **The main result.** Consider the finitely generated submodule $M = \{g_1 f_1 + \cdots + g_q f_q : g_1, \ldots, g_q \in \mathscr{E}(\Omega)\}$ of $[\mathscr{E}(\Omega)]^p$ where $f_j = (f_{1j}, \ldots, f_{pj}) \in [\mathscr{E}(\Omega)]^p$ for $1 \leqq j \leqq q$. Let $F$ be the $p \times q$ matrix $(f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$. Then $F : [\mathscr{E}(\Omega)]^q \to [\mathscr{E}(\Omega)]^p$ and $\operatorname{im}(F) = M$. In [**2**, pp. 21–25], Malgrange shows that $M = \operatorname{im}(F)$ is closed in $[\mathscr{E}(\Omega)]^p$ if each $f_{ij}$ is real analytic on $\Omega$. A zero of a function is said to be a *zero of finite order* if some derivative of the function fails to vanish there. Our main result is

THEOREM 1. *Suppose* $F = (f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$, $f_{ij} \in \mathscr{E}(\Omega)$, *and let* $r = \max\{\operatorname{rank}(F(x)) : x \in \Omega\}$. *For* $\Omega \subset \mathscr{R}^n$, *if the finitely generated submodule* $\operatorname{im}(F)$ *is closed in* $[\mathscr{E}(\Omega)]^p$, *then for every* $x \in \Omega$ *with* $\operatorname{rank}(F(x)) < r$ *there exists an* $r \times r$ *submatrix* $A$ *of* $F$ *such that* $x$ *is a zero of finite order of* $\det(A)$. *For* $\Omega \subset \mathscr{R}^1$, *the converse also holds.*

For $\Omega \subset \mathscr{R}^n$, $n > 1$, the converse fails to hold [**1**, p. 89]. For $\Omega \subset \mathscr{R}^1$, the fact that the zeros of finite order condition is sufficient follows from Malgrange's characterization of the closure of a submodule of differentiable functions [**1**, Corollary 1.7, p. 25]. For $\Omega \subset \mathscr{R}^n$, the necessity of the zeros of finite order condition can be demonstrated in the following manner. Assuming that $\operatorname{im}(F)$ is closed in $[\mathscr{E}(\Omega)]^p$, we have by the closed range theorem for Frechet spaces that $\operatorname{im}(F') = [\ker(F)]^\perp$ where $F' : [\mathscr{E}'(\Omega)]^p \to [\mathscr{E}'(\Omega)]^q$ is the transpose of $F$. Assuming that the set $Z_\infty$ of $x \in \Omega$ for which $x$ is a zero of infinite order of $\det(A)$ for every $r \times r$

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submatrix $A$ of $F$ is nonempty, we find that there exists $(T_1, \dots, T_q) \in [\ker(F)]^{\perp}$ such that for some $j$, $1 \leqq j \leqq q$, $\operatorname{supp}(T_j) = \{a\} \subset \operatorname{bd}(Z_\infty)$. Therefore $F'(S_1, \dots, S_p) = (T_1, \dots, T_q)$ for some $(S_1, \dots, S_p) \in [\mathscr{E}'(\Omega)]^p$ which leads to a distribution equation of the form $g_1 S_1 + \cdots + g_p S_p = T$ where each $g_i$ has a zero of infinite order at $a$ and $\operatorname{supp}(T) = \{a\}$, which is impossible. A complete proof of Theorem 1 will appear elsewhere.

3. **Applications.** For $F = (f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$, $f_{ij} \in \mathscr{E}(\Omega)$, define $F: [\mathscr{D}'(\Omega)]^q \to [\mathscr{D}'(\Omega)]^p$ by

$$F(S_1, \dots, S_q) = \left( \sum_{j=1}^{q} f_{1j} S_j, \dots, \sum_{j=1}^{q} f_{pj} S_j \right)$$

and let $F': [\mathscr{D}(\Omega)]^p \to [\mathscr{D}(\Omega)]^q$ be the transpose of $F$.

Suppose $P_1, \dots, P_q$ are constant coefficient linear differential operators and consider the system of variable coefficient linear differential equations

$$(1) \qquad f_{i1} P_1 S_1 + \cdots + f_{iq} P_q S_q = T_i, \qquad 1 \leqq i \leqq p,$$

where each $T_i \in \mathscr{D}'(\Omega)$. In order that there exist a solution $(S_1, \dots, S_q) \in [\mathscr{D}'(\Omega)]^q$ to (1), it is necessary that $(T_1, \dots, T_p) \in [\ker(F')]^{\perp}$ since $\operatorname{im}(F) \subset [\ker(F')]^{\perp}$. Equivalently, it is necessary that every "relation" between the rows of $(f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$ be a "relation" between $(T_1, \dots, T_p)$, that is, if $\psi_1 \boldsymbol{f}_1 + \cdots + \psi_p \boldsymbol{f}_p = (0, \dots, 0)$ where $\boldsymbol{f}_i = (f_{i1}, \dots, f_{iq})$ and $\psi_i \in \mathscr{D}(\Omega)$ for $1 \leqq i \leqq p$, then $\psi_1 T_1 + \cdots + \psi_p T_p = 0$. When is this condition also sufficient? A partition of unity argument, the closed range theorem, and Theorem 1 give

THEOREM 2. *Suppose* $F = (f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$, $f_{ij} \in \mathscr{E}(\Omega)$, *and let* $r = \max\{\operatorname{rank}(F(x)) : x \in \Omega\}$. *For* $\Omega \subset \mathscr{R}^n$, *if there exists a solution* $(S_1, \dots, S_q) \in [\mathscr{D}'(\Omega)]^q$ *to* (1) *for every* $(T_1, \dots, T_p) \in [\ker(F')]^{\perp}$, *then for every* $x \in \Omega$ *with* $\operatorname{rank}(F(x)) < r$ *there exists an* $r \times r$ *submatrix* $A$ *of* $F$ *such that* $x$ *is a zero of finite order of* $\det(A)$. For $\Omega \subset \mathscr{R}^1$, *the converse also holds.*

When can (1) be solved for every $(T_1, \dots, T_p) \in [\mathscr{D}'(\Omega)]^p$? Using the fact that $F': [\mathscr{D}(\Omega)]^p \to [\mathscr{D}(\Omega)]^q$ is one-to-one if and only if the set of $x \in \Omega$ for which $\operatorname{rank}(F(x)) = p$ is dense in $\Omega$, it is easy to see that the analog of Theorem 2 in this case involves the following condition: For every $x \in \Omega$ with $\operatorname{rank}(F(x)) < p$ there exists a $p \times p$ submatrix $A$ of $F$ such that $x$ is a zero of finite order of $\det(A)$.

Theorem 1 can also be applied to systems of variable coefficient linear differential equations of the form

$$(2) \qquad f_{i1} P_1 g_1 + \cdots + f_{iq} P_q g_q = h_i, \qquad 1 \leqq i \leqq p,$$

where each $f_{ij} \in \mathscr{E}(\Omega)$, each $P_j$ is a constant coefficient linear differential operator, and each $h_i \in \mathscr{E}(\Omega)$. In order that there exist a solution $(g_1, \ldots, g_q) \in [\mathscr{E}(\Omega)]^q$ to (2), it is necessary that $(h_1, \ldots, h_p)$ be "pointwise" in $\operatorname{im}(F)$ where $F = (f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}: [\mathscr{E}(\Omega)]^q \to [\mathscr{E}(\Omega)]^p$, that is, for each $x \in \Omega$, $T_x(h_1, \ldots, h_p) \in T_x(\operatorname{im}(F))$ where $T_x$ is the natural mapping of $[\mathscr{E}(\Omega)]^p$ onto $[\mathscr{E}(\Omega)]^p/[J_x]^p$ and $J_x$ is the ideal in $\mathscr{E}(\Omega)$ consisting of all functions in $\mathscr{E}(\Omega)$ which vanish at $x$ together with all derivatives. When is this condition also sufficient? Malgrange's characterization of the closure of a submodule of differentiable functions and Theorem 1 give

THEOREM 3. *Suppose* $F = (f_{ij})_{1 \leqq i \leqq p; 1 \leqq j \leqq q}$, $f_{ij} \in \mathscr{E}(\Omega)$, *and let* $r = \max\{\operatorname{rank}(F(x)) : x \in \Omega\}$. *For* $\Omega \subset \mathscr{R}^n$, *if there exists a solution* $(g_1, \ldots, g_q) \in [\mathscr{E}(\Omega)]^q$ *to* (2) *for every* $(h_1, \ldots, h_p)$ *which is pointwise in* $\operatorname{im}(F)$, *then for every* $x \in \Omega$ *with* $\operatorname{rank}(F(x)) < r$ *there is an* $r \times r$ *submatrix* $A$ *of* $F$ *such that* $x$ *is a zero of finite order of* $\det(A)$. *For* $\Omega \subset \mathscr{R}^1$, *the converse also holds.*

When can (2) be solved for every $(h_1, \ldots, h_p) \in [\mathscr{E}(\Omega)]^p$? It is easy to see (even without Theorem 3) that the analog of Theorem 3 in this case involves the following condition: $\operatorname{rank}(F(x)) = p$ for all $x \in \Omega$. And assuming that $\Omega \subset \mathscr{R}^n$ is $P_j$-convex for $1 \leqq j \leqq q$ (which is always the case for $\Omega \subset \mathscr{R}^1$), this condition is both necessary and sufficient.

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