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THE RADIUS OF CONVEXITY FOR A SPECIAL CLASS OF MEROMORPHIC FUNCTIONS

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Let Σ denote the class of functions $F(\zeta) = \zeta + a_0 + a_1/\zeta + \cdots$ regular in $1 < |\zeta| < \infty$. In this paper the radius of convexity for the subclass Σ_α defined by the additional condition $\operatorname{Re} F'(\zeta) > \alpha$, where $0 \leqq \alpha < 1$, is determined. The results are sharpened for functions with missing terms in the expansion. The proofs are based on inequalities for analytic functions established by the author [**3**]. The functions $F(\zeta)$ are not assumed to be schlicht; in fact, the extremal functions for $\alpha < \frac{1}{2}$ will not be schlicht. It is not known whether the univalence of $F(\zeta)$ follows from the condition $\operatorname{Re} F'(\zeta) > \frac{1}{2}$ for $R_c > |\zeta| > R > 1$. The radius of convexity ($R_c \sim 1.78$) for the class Σ with the assumption of schlichtness is due to Goluzin [**1**, p. 136]; Robertson [**2**, Theorem 4] found $R_c = 3^{1/2}$ for the subclass of schlicht and starlike functions. It will be shown that: for the class $\Sigma_{1/2}$, $R_c = 3^{1/2}$; and $R_c < 3^{1/2}$ for $\alpha > \frac{1}{2}$.

THEOREM 1. *The radius of convexity, R_0, for functions $F(\zeta) \in \Sigma_\alpha$ is given by*

$$R_0^2 \leqq \{[(3 + c)^2 + 4c]^{1/2} + (3 + c)\}/2 \tag{1}$$

where $c = 1 - 2\alpha$.

PROOF. Let

$$h(z) \equiv F'(1/z) = 1 + b_1 z^2 + \cdots. \tag{2}$$

From [**4**, Theorem 7], we have

$$\left|\frac{h'(z)}{h(z)}\right| \leqq \frac{2(1 + c)|z|}{(1 + c|z|^2)(1 - |z|^2)} \quad \text{for } |z| < 1.$$

By differentiation of (2) we obtain

$$zh'(z)/h(z) = -\zeta F''(\zeta)/F'(\zeta).$$

The condition for convexity $\operatorname{Re}(\zeta F''(\zeta)/F'(\zeta) + 1) \geqq 0$ will be satisfied if

$$2(1 + c)|z|^2 \leqq (1 + c|z|^2)(1 - |z|^2).$$

This is equivalent to $|\zeta| > R_0$.

Let $p^0(z) = (1 + cz^2)/(1 - z^2)$, then $F^0(\zeta) = \zeta + [(c+1)/2][\log(\zeta - 1)/(\zeta + 1)]$

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will have $\mathrm{Re}(\zeta F''(\zeta)/F'(\zeta) + 1) = 0$ for $\zeta = R_0$. We have $F^0(\zeta) = \zeta - (c+1)/\zeta + \cdots$.

If $c > 0$, i.e., $\alpha < \frac{1}{2}$, $|a_1| > 1$ and $F^0(\zeta)$ is not schlicht.

COROLLARY. *For the special case,* $\mathrm{Re}\, F'(\zeta) > 0$, *we have* $c = 1$ *and* $R_0 = [5^{1/2} + 2]^{1/2}$; *for* $\mathrm{Re} F'(\zeta) > \frac{1}{2}$, $c = 0$ *and* $R_0 = 3^{1/2}$.

THEOREM 2. *Let* $F(\zeta) \in \Sigma_\alpha$ *have the expansion* $F(\zeta) = \zeta + a_0 + a_n/\zeta^n + a_{n+1}/\zeta^{n+1} + \cdots$ *then the radius of convexity*

$$R_0^{n+1} = \{[(n+2+nc)^2 + 4c]^{1/2} + [n+2+nc]\}/2.$$

The proof is similar to Theorem 1, based on the inequality [**4**]

$$|h'(z)/h(z)| \leqq (1+c)n|z|^{n-1}/[1-(1-c)|z|^n - c|z|^{2n}]$$

for functions with expansion $h(z) = 1 + c_n z^n + \ldots, n \geqq 1$.

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