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BOOK REVIEWS

Foundations of Probability by Alfred Rényi. Holden-Day, Inc., San Francisco, California, 1970, xvi + 366 pp.

Probability Theory by Alfred Rényi. American Elsevier Publishing Company, Inc., New York, 1970, 666 pp.

Mathematicians must have felt highly optimistic at the beginning of the twentieth century. The foundation of most branches of mathematics had recently achieved unprecedented rigor and prospects were bright for rapid and steady progress.

It was then that D. Hilbert asserted his leadership by proposing his famous list of unsolved problems. Among the unfinished business, none was more urgent a task then the quest for a "satisfactory" theory of probability. This formidable goal was not realized until 1933, when young N. Kolmogoroff presented a solution which gained universal acceptance. His theory not only provided mathematical models for the probabilistic topics of that time, but proved to be flexible and powerful enough to adjust to most new developments and applications since then.

Among the most commendable textbooks in mathematics are some of those which attempt to present Kolmogoroff's theory to "beginners". Often the authors are eminent specialists in the subject who respond, in a wide variety of ways, to a difficult challenge. This review deals with two of the very best such books. They have many features which make them different from other textbooks, and also from one another. Both were published shortly after Rényi's untimely death which occurred on February 1, 1970.

Rényi's *Probability Theory* is a thoroughly revised and somewhat enlarged English version of his textbook in probability, published in German in 1962, and also in French and Hungarian. Its basis consists of the author's introductory lectures presented over a period of many years starting in 1948.

Foundations has a more recent origin, as Rényi explains in its preface, written in November, 1969. It started with material from a summer graduate course at Stanford University in 1966. From then on, the author decided to write a completely new book, which would have essentially no overlap with the previous one. The manuscript was completed in 1968 and is one of the most original and masterful works of its kind.

We shall proceed to review *Foundations* first. One should clarify the kind of "beginners" that Rényi had in mind. No previous knowledge of probability is assumed. However, it is essential, in order to appreciate the book, to possess a considerable amount of mathematical maturity, a good knowledge of real analysis, and a keen interest towards probability theory as a field of mathematical research. This reviewer felt, when reading many passages, a certain envy for those readers never exposed to the theory before. For instance, Rényi presents each intuitive notion with examples from the real world. Then he takes pain to persuade us that the corresponding mathematical model provided by the theory is the most natural and useful one. A stronger feeling of discovery and adventure may be realized in those readers with great enthusiasm and no prejudice.

It seems necessary to indicate in part the content of the book, in order to convey its very original organization and interesting choice of material. There are only five chapters with brief titles: 1, Experiments. 2, Probability. 3, Independence. 4, Laws of chance. 5, Dependence.

The first intuitive notion is the one of experiment. What is an experiment? Simply, it is a situation depending on chance. For instance, observing how long one has to wait for the departure of an airplane is an experiment. What is the mathematical model for an experiment? Answer: a measurable space (not a probability space as is more customary). All sets are events but measurable sets are observable events (an unusual terminology!). Why should observable events form a sigma-algebra? (After all, "observable" events in quantum mechanics do not even form an algebra.) The clarification of this point must wait until Chapter 2, where the notion of probability is introduced. However, Chapter 1 has more topics than one may suspect. The notion of random mappings is introduced here (not after that of probability, as is usual). First steps in defining independence and entropy are taken at this surprisingly early stage by introducing the notions of qualitative independence and qualitative entropies. There are also sections on Boolean algebras and on "operations with experiments".

In Chapter 2, what emerges, after an engaging discussion over the intuitive notion of probability, is not the concept of probability spaces a la Kolmogoroff, but the more general concept of Kolmogoroff-Rényi conditional probability spaces. With the more general spaces available, some situations involving unbounded measures are allowed to be kept within the frame of the theory. Rényi is especially persuasive over this choice of model. The issue does not transcend Chapter 2, however, because ordinary probability spaces are readily identified with a special case of conditional ones, and are the ones that appear almost exclusively from Chapter 3 on. However, random variables, probability distributions and expectations are presented in relation with both ordinary and conditional probability spaces.

Chapter 3 is very strong and deals with the all-important notion of stochastic independence. Rényi convinces us that qualitative independence is the basic notion because, in a certain sense, it is a necessary and sufficient condition for the construction of a probability measure for which stochastic independence is satisfied. The most important result in this direction is a special case of a theorem due to Banach for which Rényi provides a simpler proof. As a corollary, the product probability theorem for denumerable families of probability spaces follows. The last five sections of Chapter 3 deal with connections between independence and certain special topics (orthogonality, ergodic theory, information, Markov chains, etc.). They add uncommon depth to the chapter, and demand versatility from the reader.

Chapter 4 poses a paradox. How can there be laws of chance? To speak of such laws would be like speaking about "structure of chaos" or about "patterns of irregularity". Rényi manages to persuade us again: it is not in spite of the randomness of the phenomena concerned that laws of chance exist, but because of it!

Out of such a large class of topics the author picks a few conspicuous ones, starting with the original proof by Bernoulli of his law of large numbers. The more usual and short proof by using Chebyshev's inequality is also presented with its generalization to random variables. Then, the strong law of large numbers is proven for uniformly bounded random variables. The celebrated Kolmogoroff's strong law of large numbers is obtained by applying Birkhoff's ergodic theorem, whose proof is omitted. De Moivre-Laplace theorem is obtained by using the method in Bernoulli's proof, and further generalized.

The Poisson process is introduced next. Its probability space is constructed via a sequence of independent, identically distributed, exponential random variables. In this manner, it is induced by countably many random variables, as the author consistently avoids the uncountable case. The chapter ends with Lindeberg's central limit theorem, and a section on laws of fluctuation (law of the iterated logarithm, arc sin law, etc.).

Finally, Chapter 5 includes sections on laws of chance for martingales, Markov chains, stable sequences of events, mixing and exchangeable sequences. It also has a proof of Kolmogoroff consistency theorem for denumerable families of random variables.

The emphasis of *Foundations* is not in the amount of material covered, but rather in the depth in which certain representative topics are explored. Much of the material is presented in the form of exercises and problems. Rényi believes that the reader should be kept active and leaves a good amount of work for him to do. There are exactly ten exercises and ten problems, superbly selected, at the end of each chapter.

Probability Theory is also an excellent textbook but quite different in plan and emphasis. The level of knowledge and maturity demanded from the reader is somewhat lower. Discussion of some basic questions is not so sharp, but many topics with little or no coverage in *Foundations* are treated thoroughly here. Both books have some similarities also, but mostly derived from the unmistakably Rényian presence as a teacher and researcher.

The first two chapters deal with probability spaces. Conditional probability spaces are also introduced. The general theory of random variables (Chapter 4) is preceded by a long chapter on the discrete case. A chapter on dependence is followed by an important chapter on characteristic functions of random vectors. While the method of characteristic functions is applied very little in *Foundations*, it is given full treatment in *Probability Theory*. The chapters on limit theorems make more use of that method. There is a long appendix on information theory. There are, of course, many more exercises than in *Foundations* and also more examples related with applications.

Both books complement each other well and have, as said before, little overlap. They represent nearly opposite approaches to the question of how the theory should be presented to beginners. Rényi excels in both approaches. *Probability Theory* is an imposing textbook. *Foundations* is a masterpiece.

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A comprehensive introduction to differential geometry, Volumes I and II, by Michael Spivak. Brandeis University, 1969.

The following is a review of volume I of Spivak's book and about half a review of volume II. In a subsequent issue of the BULLETIN I would like to say more about volume II. (I hope that volume III, now in the works, will by then have appeared.)

In the introduction "How this book came to be," Spivak makes the following remark, which I endorse (with some reservations, which I will try to spell out below). "Today a dilemma confronts any one intent on penetrating the mysteries of differential geometry. On the one hand one can consult numerous classical treatments of the subject in an attempt to form some idea of how the concepts within it developed."

"Unfortunately a modern mathematical education tends to make classical mathematical works unaccessible, particularly those in differential geometry. On the other hand one can now find texts as modern in spirit and as clean in exposition as Bourbaki's algebra. But a thorough study of these books usually leaves one unprepared to consult classical works, and entirely ignorant of the relationship between elegant modern constructions and their classical counterparts. Most students eventually find that this ignorance of the roots of a subject has its price—no one denies that modern definitions are clear, elegant, and precise; it is just that it is