so sharp, but many topics with little or no coverage in *Foundations* are treated thoroughly here. Both books have some similarities also, but mostly derived from the unmistakably Rényian presence as a teacher and researcher.

The first two chapters deal with probability spaces. Conditional probability spaces are also introduced. The general theory of random variables (Chapter 4) is preceded by a long chapter on the discrete case. A chapter on dependence is followed by an important chapter on characteristic functions of random vectors. While the method of characteristic functions is applied very little in *Foundations*, it is given full treatment in *Probability Theory*. The chapters on limit theorems make more use of that method. There is a long appendix on information theory. There are, of course, many more exercises than in *Foundations* and also more examples related with applications.

Both books complement each other well and have, as said before, little overlap. They represent nearly opposite approaches to the question of how the theory should be presented to beginners. Rényi excels in both approaches. *Probability Theory* is an imposing textbook. *Foundations* is a masterpiece.

ALBERTO R. GALMARINO

A comprehensive introduction to differential geometry, Volumes I and II, by Michael Spivak. Brandeis University, 1969.

The following is a review of volume I of Spivak's book and about half a review of volume II. In a subsequent issue of the BULLETIN I would like to say more about volume II. (I hope that volume III, now in the works, will by then have appeared.)

In the introduction "How this book came to be," Spivak makes the following remark, which I endorse (with some reservations, which I will try to spell out below). "Today a dilemma confronts any one intent on penetrating the mysteries of differential geometry. On the one hand one can consult numerous classical treatments of the subject in an attempt to form some idea of how the concepts within it developed."

"Unfortunately a modern mathematical education tends to make classical mathematical works unaccessible, particularly those in differential geometry. On the other hand one can now find texts as modern in spirit and as clean in exposition as Bourbaki's algebra. But a thorough study of these books usually leaves one unprepared to consult classical works, and entirely ignorant of the relationship between elegant modern constructions and their classical counterparts. Most students eventually find that this ignorance of the roots of a subject has its price—no one denies that modern definitions are clear, elegant, and precise; it is just that it is impossible to comprehend how any one ever thought of them."

He then goes on to say that to understand the roots one should not turn one's back on the modern point of view that has been so painstakingly acquired. The main enterprise should be to mediate between the two points of view.

This should be done with a little circumspection. "Of course I do not think one should follow all the intricacies of the historical process with its inevitable duplications and false leads. What is intended, rather, is a presentation of the subject along the lines which its development might have followed; as Bernard Morin said to me, there is no reason, in mathematics any more than in biology, why ontogeny must recapitulate phylogeny. When modern terminology finally is introduced it should be as an outgrowth of this (mythical) historical development."

The remarks above are a prologue to what turns out to be, not volume I and volume II of one book, but two quite dissimilar books. Book I is an introduction (and I think a superb one) to the fundamentals of modern differential geometry. Except for Lang's book on manifolds I cannot think of another differential geometry text that does the foundations so well. (I will discuss it in more detail below.)

Volume II could be described, I suppose, as a treatise on classical differential geometry; however, it is a highly unconventional one. It begins with the classical theory of curves and surfaces, followed by a discussion of Gauss's "Disquisitiones" and an analysis of his contribution to surface theory. (A marvelous section is included here: "How to read Gauss.") This part of the book ends with a modern treatment of surface theory using exp's and the notation of volume I.

The second part of the book is on classical Riemannian geometry. It begins appropriately enough with Riemann's "Ueber die Hypothesen, welche der Geometrié zur Grunde liegen" reproduced in its entirety; and is followed by a section of commentary on "What Riemann said," a delightful introduction to classical Riemannian geometry. The rest of the book is devoted to the theory of connections, beginning with the Levi-Civita connection (defined in terms of Christoffel symbols) then going on to the modern theory of connections on vector bundles, then back again to E. Cartan and the repére mobile, and then forward again to the modern theory of connections on principal bundles. The translations between the various points of view are meticulously spelled out.

I must confess to certain misgivings about volume II. As Spivak points out in his introduction, it is highly desirable to explain a concept not just by explaining what it means but by also explaining how it occurred to people to think of it. However, a rather low brow concept may have a rather formidable history, in which case one wonders whether such an explanation is all that helpful. Take the notion of connection for instance. That the Levi-Civita connection describes so elegantly the geometry of a Riemannian manifold (or that it exists at all) is more or less an accident (like the fact that the Kähler structure so adequately describes the geometry of a nonsingular projective variety). There are, as far as I know, only a handful of connections like the Levi-Civita connection. Usually one hoaks up a connection on a vector bundle or principal bundle by a partition of unity; and all one expects to get is a way of differentiating globally. To say that a connection is a "way of differentiating globally" is, I feel, an adequate description of it, acceptable even to a fastidious graduate student.

Notwithstanding, I found Spivak's guided tour of 19th century geometry highly illuminating. To a student unfamiliar with the sources, this seems to be as good a place as any to get acquainted with differential geometry's Great Tradition.

To get back to volume I, here is a brief summary of the contents, chapter by chapter.

Chapter 1 is a discussion of topological manifolds. It is mainly a survey of the standard examples S^n, T^n, P^n etc. It contains a section I particularly liked on the various ways of immersing projective 2-space in R^3, e.g. attaching a disk to a Moebius band, cross caps, the image of a monoidal transformation, etc.

Chapter 2 defines differentiable manifolds and talks about immersions.

Chapter 3 is a formidable chapter (which I disliked), on the tangent bundle. All the various definitions of a tangent bundle are given: via derivations of functions, via equivalence classes of curves, via charts etc.; and it is shown that any functor from manifolds to vector bundles which assigns df to R^n *is* the tangent bundle. The proof is quite long and involves two hideous commutative diagrams. Well, okay.

Chapter 4 is on tensors. (There is a helpful discussion of the classical versus the modern point of view on pp. 7–8 and also a discussion of how the perverse terminology "covariant" and "contravariant" came to be.)

Chapters 5 and 6 are on vector fields and foliations, respectively. Chapter 5 begins by reviewing some of the problems that can occur when one tries to integrate a differential equation (on R^1 say). My tendency, when I teach this, a deplorable one, is to sweep all these problems under the carpet. Then the Picard existence theorem is proved, and the one parameter group constructed. The Lie derivative L_ξ (of a form, tensor, vector field) is defined, and it is carefully shown that $L_\xi \eta = [\xi, \eta]$. There is an excellent discussion at the end of the section, much clearer than I have seen in any other standard text, of the parallelogram law for computing $[\xi, \eta]$.

Chapter 6 begins with a discussion of two-dimensional distributions in R^3, and when integral manifolds exist for them. The integrability condition is derived in such a way that it is clear how to generalize it, and then the Frobenius theorem is proved (both the local and global versions).

Chapters 7 and 8 are on forms and integration. They contain the Poincaré lemma, Stokes' theorem and the proof that for an oriented connected n-manifold $\int : H^n_{\text{compact}} \to R$ is an isomorphism. The proof of the last assertion, which depends on a lemma that Spivak calls "integration in polar coordinates" is as elegant a proof as I have seen anywhere.

Chapter 9 is on Riemannian geometry. The equation for geodesics are derived in their local form (with Christoffel symbols), the exp map is defined, Gauss's lemma proved, and one global result deduced from it, the Hopf-Rinow theorem. One nice piece of motivational detail: The technique of first variation (Euler's equation) is described for a general variational problem before being applied to the variation of arc length; and, as an example, the problem of "minimal surfaces of revolution" is worked out.

Chapter 10 is a clean presentation of the elementary facts about Lie groups, going as far as the closed subgroup theorem.

Finally, *Chapter* 11 is a short treatment of algebraic topology via de Rham theory. Using Mayer-Vietoris, Spivak proves that the cohomology of a compact manifold is finite dimensional, proves the excision theorem, Poincaré duality, the Thom isomorphism, etc. Together with Chapters 7 and 8 this chapter provides a very nice introduction to differential topology.

Two concluding remarks: The problems sets make up about a third of the book. The problems are often quite ingenious and clever; however, some of them are too hard for a beginning student! The illustrations, though rough, are awfully good, and there are many of them. They are an invaluable aid to understanding the text.

VICTOR GUILLEMIN