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ON A QUESTION OF DOUGLAS AND FILLMORE

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Let $\mathscr{K}$ denote the ideal of compact operators in $\mathscr{B}(H)$, the bounded linear operators on a Hilbert space H, and let ν denote the canonical homomorphism from $\mathscr{B}(H)$ onto the Calkin algebra $\mathscr{B}(H)/\mathscr{K}$. Brown, Douglas, and Fillmore [**2**], in an elegant and interesting paper, showed that if $\nu(T)$ is normal, if $\sigma_e(T) \equiv \sigma(\nu(T))$ is homeomorphic to a finite graph, and if the Fredholm index $i(T-\lambda) = 0$ for λ in the holes of $\sigma_e(T)$, then T is the sum of a normal operator and a compact operator. It thus becomes natural to ask whether a similar analysis can be carried out when $\sigma_e(T)$ has positive area. Let us consider the simplest case. If $\sigma_e(T) = \Delta$, the closed unit disc, and if $\nu(T)$ is normal, is T of the form normal plus compact? To indicate the limited scope of our knowledge in this area it is not even known whether the operator $S \oplus M$ is normal plus compact, where S is the unilateral shift and M is multiplication by z on $L^2(\Delta, dm)$. In fact R. G. Douglas [**4**, p. 62] raises this specific question, which had earlier been broached by P. A. Fillmore.

In this note we will give an affirmative answer to the Douglas–Fillmore question. Unfortunately, the techniques employed here do not shed much light on the general problem.

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THEOREM. *The operator $S \oplus M$ is of the form normal plus compact.*

We begin by obtaining a matrix representation for M on $L^2(\Delta)$. We do this in the following rather roundabout manner: Let A^2 denote the analytic functions in $L^2(\Delta)$. Then A^2 is a Hilbert space and has as an orthonormal basis $\{[(n+1)/\pi]^{1/2}z^n\}_{n=0}^{\infty}$ [**5**, p. 15]. Clearly A^2 is invariant for M and $B = M|A^2$ is easily seen to be unitarily equivalent to the weighted shift S_1 with weights $\{[n/(n+1)]^{1/2}\}_{n=1}^{\infty}$. Since M is the minimal normal extension of the subnormal operator B, the characterization of the minimal normal extension of a subnormal weighted shift obtained in [**6**] yields the following matrical description of M. The operator M is unitarily equivalent to the matrix

$$N = \begin{bmatrix} S_1 & D_2 & & \mathbf{0} \\ & S_2 & D_3 & \\ & & S_3 & \ddots \\ & \mathbf{0} & & \ddots \end{bmatrix}$$

where S_n is the weighted shift with weights

$$\left\{ a_k^{(n)} = \left(\frac{kk}{(n+k-1)(n+k)} \right)^{1/2} \right\}_{k=1}^{\infty}$$

and D_n is the diagonal operator with entries

$$\left\{ b_k^{(n)} = \left(\frac{(n-1)(n-1)}{(n+k-2)(n+k-1)} \right)^{1/2} \right\}_{k=1}^{\infty}.$$

To prove this one need only verify the identities $(b_k^{(n+1)})^2 = (a_k^{(n)})^2 + (b_k^{(n)})^2 - (a_{k-1}^{(n)})^2$ and $a_k^{(n+1)}b_k^{(n+1)} = a_k^{(n)}b_{k+1}^{(n+1)}$ for $k, n = 1, 2, \ldots$ ($a_0^{(n)} = 0$ by definition) from [**6**].

If we can show that there exists a compact operator K' such that $N + K'$ is unitarily equivalent to $S \oplus N$, then we will have shown that there exists a compact operator K such that $M + K$ is unitarily equivalent to $S \oplus M$, thus answering the Douglas–Fillmore question. The existence of such a K' follows from the following facts:

(I) There exists a compact operator K_1 such that $S_1 + K_1 = S$. (Let

K_1 be the weighted shift with weights $\{1 - [n/(n+1)]^{1/2}\}_{n=1}^{\infty}$ [**5**, p. 86].)

(II) D_2 is a compact operator, in fact each D_n is compact [**5**, p. 86].

(III) $S_n - S_{n+1}$ and $D_n - D_{n+1}$ are compact for each $n = 1, 2, 3, \ldots$, and $\|S_n - S_{n+1}\|$, $\|D_n - D_{n+1}\| < 1/n$ for each $n = 1, 2, \ldots$. (Here $S_n - S_{n+1}$ is the weighted shift with weights $\{c_k^{(n)} = a_k^{(n)} - a_k^{(n+1)}\}_{k=1}^{\infty}$ and $D_n - D_{n+1}$ is the diagonal operator with entries $\{d_k^{(n)} = b_k^{(n)} - b_k^{(n+1)}\}_{k=1}^{\infty}$. Straightforward calculations reveal that $c_k^{(n)}, d_k^{(n)} \to 0$ as $k \to \infty$ and that $|c_k^{(n)}|, |d_k^{(n)}| < 1/n$ for all $n, k = 1, 2, \ldots$. The first fact shows that $S_n - S_{n+1}$ and $D_n - D_{n+1}$ are compact, while the second fact shows that $\|S_n - S_{n+1}\|$, $\|D_n - D_{n+1}\| < 1/n$.)

Facts (I), (II), and (III) imply that if

$$K' = \begin{bmatrix} K_1 & -D_2 & & & \\ & S_1 - S_2 & D_2 - D_3 & & \mathbf{0} \\ & & S_2 - S_3 & D_3 - D_4 & \\ & \mathbf{0} & & S_3 - S_4 & \ddots \\ & & & & \ddots \end{bmatrix}$$

then K' is a compact operator, and $N + K'$ is unitarily equivalent to $S \oplus N$. Hence there exists a compact operator K such that $S \oplus M$ is unitarily equivalent to $M + K$.

REMARKS. (1) Since $S \oplus M$ is normal plus compact, there exists a normal operator N and a compact operator K such that the residual spectrum of $N + K$ is an open set.

(2) Since the self-commutator of $S \oplus M$ is trace class with nonzero trace, it is impossible to write $S \oplus M$ as normal plus trace class.

(3) Using Berg's theorem [**1**] one can easily verify that $S \oplus M$ is normal plus compact, where S is a unilateral shift of finite multiplicity and M is any normal operator whose spectrum contains Δ.

(4) Let R be a normal operator whose spectrum is $\Delta \cap \{z : \mathrm{Re}(z) \geqq 0\}$. Then $S \oplus R$ is an example of a nonquasitriangular operator whose square is normal plus compact and hence quasitriangular [see **3**].

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