BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, March 1973

FUNCTIONS WITH A SPECTRAL GAP

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Communicated by Gian-Carlo Rota, September 13, 1972

Introduction. In harmonic analysis, it is important to know how various properties of a function on $\boldsymbol{R}^n$ reflect themselves as restrictions on its *spectrum*, i.e., the support of its (distributional) Fourier transform. Thus, according to Paley and Wiener, a compact spectrum is characteristic of entire functions of exponential type. In this note we consider a milder restriction: it is only required of the spectrum that it be smaller than the whole n-space. Our results extend those of Levinson, Logan, Ehrenpreis and Malliavin; cf. also Boas [**1**]. Here we give only bare outlines of proofs; we employ standard vector notations: $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots x_n)$ are points of $\boldsymbol{R}^n$ and (t, x) denotes $\sum_1^n t_j x_j$; $|t| = (t, t)^{1/2}$, and dt denotes Haar measure on $\boldsymbol{R}^n$.

1. A *gap* in a distribution on $\boldsymbol{R}^n$ is a nonvoid open ball disjoint from its support. A *spectral gap* in a tempered distribution is a gap in its Fourier transform. In particular, an L^1 function f has a spectral gap if its Fourier transform $\hat{f}(x)$ vanishes on some nonvoid open set. Such f cannot decay too rapidly, by virtue of the following result of N. Levinson.

THEOREM A. *Let $f \in L^1(\boldsymbol{R})$, and suppose for some $\delta > 0$*

$$\int_0^\infty |f(t)| e^{\delta t}\, dt < \infty. \tag{1}$$

Then, if $\hat{f}(x)$ vanishes throughout any interval, it vanishes identically.

For the proof, one need only check [**4**, p. 74] that (1) implies that $\hat{f}(x)$ is the boundary value of a function holomorphic in a strip above the real axis. (Actually Levinson, *loc. cit.*, proves much deeper results, with (1) replaced by weaker hypotheses that do not force analyticity of $\hat{f}(x)$. An account of these, based on a new and simple method, will be given by me in a subsequent paper. The weaker Theorem A will serve as a basis for the present discussion.)

Theorem A admits a straightforward generalization to $\boldsymbol{R}^n$. Let us say that a convex cone K in $\boldsymbol{R}^n$ (all cones will be supposed to have vertex at the origin) is *minor* if there exists a unit vector $t^0 \in \boldsymbol{R}^n$ such that $\inf(t^0, t)$; $t \in K, |t| = 1$, is positive. Thus, a half-line in $\boldsymbol{R}^1$, or a sector of opening

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AMS(MOS) subject classifications (1970). Primary 42A68, 42A76; Secondary 31B05.
Key words and phrases. Spectrum, support, Fourier transform, Poisson integral.

less than π in $\boldsymbol{R}^2$, are minor. (It is easy to see that an open convex cone K is minor if and only if there exists a nonsingular linear transformation of $\boldsymbol{R}^n$ carrying K onto the "first quadrant", i.e., the set K^+ of points of $\boldsymbol{R}^n$ having all coordinates positive.) By an analyticity argument, as above, one proves easily

THEOREM A′. *Let f be a tempered function on $\boldsymbol{R}^n$ and suppose for some minor cone K and $\delta > 0$*

$$\int_{\boldsymbol{R}^n \setminus K} |f(t)| e^{\delta |t|} \, dt < \infty.$$

Then, if f has a spectral gap, $f = 0$.

The condition that K be minor is essential, since for $n \geqq 2$ there exist nontrivial $f \in L^1(\boldsymbol{R}^n)$ which vanish on a half-space and have a spectral gap [**9**, p. 172].

If $f \in L^1(\boldsymbol{R}^n)$ does not vanish identically, any $\phi \in L^\infty(\boldsymbol{R}^n)$ satisfying the convolution equation $f * \phi = 0$ has a spectral gap. Hence it is easy, by duality, to deduce from Theorem A′ the following approximation theorem, as observed recently for $n = 1$ by D. J. Newman [**6**]:

Let K be a minor cone in $\boldsymbol{R}^n$, $\delta > 0$, and $w(t)$ a nonnegative measurable function on $\boldsymbol{R}^n$ equal to 1 *on K, and satisfying*

$$\int_{\boldsymbol{R}^n \setminus K} w(t) e^{\delta |t|} \, dt < \infty.$$

Then, for any $f \in L^1(\boldsymbol{R}^n)$ not identically zero, the translates of f span $L^1(w\,dt)$.

In particular, the translates of f, restricted to K, span the integrable functions on K.

2. Logan, in a 1965 dissertation [**5**, p. 26, Theorem 5.2.1] proved

THEOREM B. *Let $f \in L^\infty(\boldsymbol{R})$ be nonnegative on $\boldsymbol{R}^+$. Then, if f has a spectral gap containing* 0, *f vanishes identically.*

Observe that here (and in the next section) the *position* of the spectral gap (i.e. containing the origin) is essential. We sketch a proof, based on a new idea which suggests the correct generalization to $\boldsymbol{R}^n$. We may assume $f \in L^1(\boldsymbol{R})$ (for to reduce the general case to this, consider $f(t) \cdot (\sin \varepsilon t)^2/t^2$ with sufficiently small ε), and that $\hat{f}(t) = 0$ for $|x| \leqq 3$. A simple application of Parseval's formula gives for $m = 0, 1, \ldots$

$$2\pi \int_0^\infty f(t) t^m e^{-t} \, dt = m! \int_{-\infty}^\infty \hat{f}(x)(1 - ix)^{-m-1} \, dx.$$

The integral on the right is bounded by $(\int_{|x| \geqq 3} |x|^{-m-1} dx) \cdot \|\hat{f}\|_\infty = O(3^{-m})$, hence

$$\int_0^\infty f(t)e^t\,dt = \int_0^\infty f(t)\left(\sum_{m=0}^\infty (2t)^m/m!\right) e^{-t}\,dt < C\sum_{m=0}^\infty (2/3)^m < \infty.$$

Now Theorem A implies $f \equiv 0$. Q.E.D.

Let K, K' be closed cones in $\boldsymbol{R}^n$; we say K' is *strongly enclosed* by K if $\{x \in K' : |x| = 1\}$ is in the interior of K. We now state our first main result:

THEOREM B′. *Let f be a tempered function on $\boldsymbol{R}^n$ having a spectral gap containing* 0, *and nonnegative a.e. on the closed convex cone K. Let K' be any closed cone strongly enclosed by K. Then, for some $\delta = \delta(f; K') > 0$, $\int_K f(t)e^{\delta|t|}\,dt < \infty$.*

COROLLARY. *In the hypotheses of Theorem* B′, *if the cone complementary to K is minor, then f vanishes identically.*

The proof of Theorem B′ is in principle like that sketched for Theorem B, but complicated technically. The Corollary then follows using Theorem A′.

REMARK. The hypothesis of nonnegativity on K can be weakened to having range in a sector of opening less than π.

3. Logan (*loc. cit.*) also established a relation between a spectral gap about 0 and exponential decay of the Poisson integral [**5**, Theorems 6.2.3 and 6.3.1]:

THEOREM C. *Let $f \in L^\infty(\boldsymbol{R})$. The spectrum of f is disjoint from $(-a, a)$ if and only if the Poisson integral*

$$u(x; y) = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\frac{y}{(x-\xi)^2 + y^2}\,d\xi$$

of f satisfies

$$|u(x; y)| \leqq Ae^{-ay}, \qquad y > 0, \tag{2}$$

where A is a positive constant independent of x.

This theorem readily implies that of Paley and Wiener. Logan's proof uses analytic functions. I gave [**8**, p. 152] a proof using only Fourier analysis, which can be extended to n dimensions. Letting $B(x^0; a)$ denote the open ball in $\boldsymbol{R}^n$ with center x^0 and radius a, we have our second main result:

THEOREM C′. *Let f be a locally integrable function on $\boldsymbol{R}^n$ such that*

$$\int (1 + |x|^2)^{-(n+1)/2}\,|f(x)|\,dx < \infty. \tag{3}$$

The spectrum of f is disjoint from $B(0; a)$ if and only if the Poisson integral

$$u(x;y) = c_n \int_{\boldsymbol{R}^n} f(\xi) \frac{y}{(|x-\xi|^2 + y^2)^{(n+1)/2}} \, d\xi$$

of f satisfies, for every $\varepsilon > 0$,

$$\int |u(x,y)|(1+|x|^2)^{-(n+1)/2} \, dx \leqq A(\varepsilon) e^{-(a-\varepsilon)y}, \qquad y > 0, \tag{4}$$

where $A(\varepsilon)$ is a positive number independent of x. If (3) *is replaced by the stronger condition $f \in L^\infty$,* (4) *is to be replaced by*

$$|u(x,y)| \leqq A(\varepsilon) e^{-(a-\varepsilon)y}, \qquad y > 0. \tag{4a}$$

The proof requires estimates for "minimal extrapolations" from the interior, as well as the exterior, of a ball; these will be given elsewhere. As in the case $n = 1$, the "only if" part of the theorem can be strengthened when $f \in L^\infty$.

If, for $k \in L^1(\boldsymbol{R}^n)$, we denote by $k_{(y)}$ the "dilated function":

$$k_{(y)}(x) = y^{-n} k(y^{-1}x); \qquad x \in \boldsymbol{R}^n, y > 0,$$

then Theorem C′ (in the case $f \in L^\infty$) may be written: *Let*

$$k(t) = c_n (1 + |t|^2)^{-(n+1)/2} \tag{5}$$

(so that $\hat{k}(x) = e^{-|x|}$); the condition

$$|(f * k_{(y)})(x)| \leqq A(\varepsilon) \hat{k}((a-\varepsilon)y), \qquad y > 0, \tag{6}$$

holds for all $\varepsilon > 0$, if and only if the spectrum of f is disjoint from $B(0;a)$.

Now, this proposition can be established for a large class of kernels $k(t)$ in place of (5), using exactly the same method; *in particular, for* $k(t) = e^{-|t|^2}$, a result obtained otherwise by Ehrenpreis and Malliavin; see [**3**, Corollary 5]. With this special choice of k, we may permit f in (6) to be any tempered distribution.

4. Assuming (4a) holds for a *single value* of x, we can nonetheless obtain spectral information about f. First, some notation: a locally integrable function on $\boldsymbol{R}^n$ is *anti-radial* if its integral over $B(0;r)$ vanishes for every $r > 0$. Every locally integrable function admits an essentially unique decomposition into a radial and an anti-radial part (for $n = 1$, this is just the even–odd decomposition). We now state our third main result:

THEOREM D. *Let f be a bounded measurable function on $\boldsymbol{R}^n$ whose Poisson integral $u(x;y)$ satisfies $|u(0;y)| \leqq Ae^{-ay}$. Then, the radial part of f has spectrum disjoint from $B(0;a)$.*

Combining Theorems C′ and D we deduce a proposition solely about harmonic functions: *If u is the Poisson integral of a radial function in*

$L^\infty(\boldsymbol{R}^n)$, (4a) *holds for every* $x \in \boldsymbol{R}^n$ *if it holds for* $x = 0$. For $n = 1$ this is a consequence of a classical Phragmén-Lindelöf theorem, but for $n > 1$ it appears to be new. Another corollary of Theorem D is: *If u is a bounded harmonic function on $\boldsymbol{R}^n \times \boldsymbol{R}^+$ satisfying $u(0; y) = O(e^{-ay})$ as $y \to +\infty$, for every $a > 0$, then $u(0, y)$ vanishes identically* (*or, what is the same thing, $u(x; 0+)$ is an anti-radial function on $\boldsymbol{R}^n$*).

Theorem D also yields a particularly simple proof of the existence of "lacunae" for the wave equation (cf. [**3**, p. 417] for terminology). Also, *the analog of Theorem* D *for the kernel* $k(t) = e^{-|t|^2}$ *is valid*; this is a refinement of a theorem in [**3**].

Proof of Theorem D. Assume first $f \in L^1(\boldsymbol{R}^n)$. We may assume f radial, since the anti-radial part contributes nothing to $u(0; y)$. By assumption,

$$\left| \int_{\boldsymbol{R}^n} f(x) \cdot y(|x|^2 + y^2)^{-(n+1)/2} \, dx \right| \leqq Ce^{-ay}.$$

Substituting here

$$y(|x|^2 + y^2)^{-(n+1)/2} = A_n \int e^{-y|t|} \cdot e^{-i(t,x)} \, dt$$

(where A_n depends only on n), and applying Fubini's Theorem, yields (integrations are over $\boldsymbol{R}^n$):

$$A_n \left| \int \hat{f}(t) \cdot e^{-y|t|} \, dt \right| \leqq Ce^{-ay}.$$

Writing $\hat{f}(t) = \phi(|t|)$, we have

$$\int_0^\infty s^{n-1} \phi(s) e^{-ys} \, ds = O(e^{-ay}), \qquad y \to +\infty.$$

Now a simple argument shows that $\phi(s)$ must vanish for $s < a$, hence $\hat{f}(t) = 0$ for $|t| < a$, as we wished to show.

The general case, when f need not belong to L^1, leads to serious complications; to be able to perform the crucial "Fubini" step, we replace the Fourier kernel $e^{-i(t,x)}$ by that of Bochner [**2**, p. 112, (5)], in an n-dimensional version, and then suitably extend the spectral analysis of Pollard [**7**]. (This is also applicable to f which satisfy only (3).)

5. Let G denote any l.c.a. group, $\hat{G}$ its dual. Let E, $\hat{E}$ be closed subsets of G, $\hat{G}$ respectively. We say the pair $(E, \hat{E})$ is *interpolatory* if, for every $f \in L^1(G)$, there exists $f_0 \in L^1(G)$ supported in E such that $\hat{f}_0(x) = \hat{f}(x)$, $x \in \hat{E}$. This is equivalent to saying that every function in $L^1(G \backslash E)$ extends to an element of $L^1(G)$ whose Fourier transform vanishes on $\hat{E}$. For instance, one can show when $G = \boldsymbol{R}^n$, that *this is the case if $\boldsymbol{R}^n \backslash E$ and $\hat{E}$ are compact.* On the other hand, Theorem B′ implies that $(E, \hat{E})$ is not inter-

polatory if $\boldsymbol{R}^n\backslash E$ contains a nonvoid open cone and $\hat{E}$ has interior. Thus, the Fourier transforms of functions supported on proper subcones of $\boldsymbol{R}^n$ are constrained in their local behavior, they possess "local structure". The detailed nature of this local structure was somewhat clarified in [**9**] for the analogous situation in $L^\infty(\boldsymbol{R}^n)$. For $n = 1$ one can show that the presence of arbitrarily long intervals in the complement of the spectrum already forces local structure.

5. **Acknowledgement.** I wish to thank Yngve Domar for valuable criticisms of an earlier draft of this paper. A fuller account of the many variants and improvements he suggested must wait for our detailed version of the present material.

Bibliography

1. R. P. Boas, Jr., *Some theorems on Fourier transforms and conjugate trigonometric integrals*, Trans. Amer. Math. Soc. **40** (1936), 287–308.

2. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932.

3. Leon Ehrenpreis, *Fourier analysis on nonconvex sets*, Ist. Naz. Alta Mat. Sympos. Math. **7** (1971), 413–419.

4. Norman Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ., vol. 26, Amer. Math. Soc., Providence, R.I., 1940. MR **2**, 180.

5. Benjamin F. Logan, Jr., *Properties of high-pass signals*, Dissertation presented to the Electrical Engineering Faculty of Columbia University, New York, 1965.

6. D. J. Newman, *Translates are always dense on the half line*, Proc. Amer. Math. Soc. **21** (1969), 511–512. MR **38** #4904.

7. H. Pollard, *The harmonic analysis of bounded functions*, Duke Math. J. **20** (1953), 499–512. MR **15**, 215.

8. H. S. Shapiro, *Topics in approximation theory*, Lecture Notes in Math., vol. 187, Springer-Verlag, Berlin and New York, 1971.

9. ———, *Bounded functions with one-sided spectral gaps*, Michigan Math. J. **19** (1972), 167–172.

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