FUNCTIONS WITH A SPECTRAL GAP

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Introduction. In harmonic analysis, it is important to know how various properties of a function on $\mathbb{R}^n$ reflect themselves as restrictions on its spectrum, i.e., the support of its (distributional) Fourier transform. Thus, according to Paley and Wiener, a compact spectrum is characteristic of entire functions of exponential type. In this note we consider a milder restriction: it is only required of the spectrum that it be smaller than the whole $n$-space. Our results extend those of Levinson, Logan, Ehrenpreis and Malliavin; cf. also Boas [1]. Here we give only bare outlines of proofs; we employ standard vector notations: $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots, x_n)$ are points of $\mathbb{R}^n$ and $(t, x)$ denotes $\sum^n_{i=1} t_j x_j$, $|t| = (t, t)^{1/2}$, and $dt$ denotes Haar measure on $\mathbb{R}^n$.

1. A gap in a distribution on $\mathbb{R}^n$ is a nonvoid open ball disjoint from its support. A spectral gap in a tempered distribution is a gap in its Fourier transform. In particular, an $L^1$ function $f$ has a spectral gap if its Fourier transform $\hat{f}(x)$ vanishes on some nonvoid open set. Such $f$ cannot decay too rapidly, by virtue of the following result of N. Levinson.

**Theorem A.** Let $f \in L^1(\mathbb{R})$, and suppose for some $\delta > 0$

\[ \int_{-\infty}^{\infty} |f(t)| e^{\delta t} dt < \infty. \]

Then, if $\hat{f}(x)$ vanishes throughout any interval, it vanishes identically.

For the proof, one need only check [4, p. 74] that (1) implies that $\hat{f}(x)$ is the boundary value of a function holomorphic in a strip above the real axis. (Actually Levinson, loc. cit., proves much deeper results, with (1) replaced by weaker hypotheses that do not force analyticity of $\hat{f}(x)$.) An account of these, based on a new and simple method, will be given by me in a subsequent paper. The weaker Theorem A will serve as a basis for the present discussion.

Theorem A admits a straightforward generalization to $\mathbb{R}^n$. Let us say that a convex cone $K$ in $\mathbb{R}^n$ (all cones will be supposed to have vertex at the origin) is minor if there exists a unit vector $t^0 \in \mathbb{R}^n$ such that $\inf(t^0, t); t \in K, |t| = 1$, is positive. Thus, a half-line in $\mathbb{R}^1$, or a sector of opening...
less than \( \pi \) in \( \mathbb{R}^2 \), are minor. (It is easy to see that an open convex cone \( K \) is minor if and only if there exists a nonsingular linear transformation of \( \mathbb{R}^n \) carrying \( K \) onto the “first quadrant”, i.e., the set \( K^+ \) of points of \( \mathbb{R}^n \) having all coordinates positive.) By an analyticity argument, as above, one proves easily

**Theorem A'.** Let \( f \) be a tempered function on \( \mathbb{R}^n \) and suppose for some minor cone \( K \) and \( \delta > 0 \)

\[
\int_{\mathbb{R}^n \setminus K} |f(t)| e^{\delta |t|} \, dt < \infty.
\]

Then, if \( f \) has a spectral gap, \( f = 0 \).

The condition that \( K \) be minor is essential, since for \( n \geq 2 \) there exist nontrivial \( f \in L^1(\mathbb{R}^n) \) which vanish on a half-space and have a spectral gap [9, p. 172].

If \( f \in L^1(\mathbb{R}^n) \) does not vanish identically, any \( \phi \in L^\infty(\mathbb{R}^n) \) satisfying the convolution equation \( f \ast \phi = 0 \) has a spectral gap. Hence it is easy, by duality, to deduce from Theorem A' the following approximation theorem, as observed recently for \( n = 1 \) by D. J. Newman [6]:

Let \( K \) be a minor cone in \( \mathbb{R}^n \), \( \delta > 0 \), and \( w(t) \) a nonnegative measurable function on \( \mathbb{R}^n \) equal to 1 on \( K \), and satisfying

\[
\int_{\mathbb{R}^n \setminus K} w(t) e^{\delta |t|} \, dt < \infty.
\]

Then, for any \( f \in L^1(\mathbb{R}^n) \) not identically zero, the translates of \( f \) span \( L^1(w \, dt) \).

In particular, the translates of \( f \), restricted to \( K \), span the integrable functions on \( K \).

2. Logan, in a 1965 dissertation [5, p. 26, Theorem 5.2.1] proved

**Theorem B.** Let \( f \in L^\infty(\mathbb{R}) \) be nonnegative on \( \mathbb{R}^+ \). Then, if \( f \) has a spectral gap containing 0, \( f \) vanishes identically.

Observe that here (and in the next section) the position of the spectral gap (i.e. containing the origin) is essential. We sketch a proof, based on a new idea which suggests the correct generalization to \( \mathbb{R}^n \). We may assume \( f \in L^1(\mathbb{R}) \) (for to reduce the general case to this, consider \( f(t) \cdot (\sin \varepsilon t)^2 / t^2 \) with sufficiently small \( \varepsilon \)), and that \( \hat{f}(t) = 0 \) for \( |x| \leq 3 \). A simple application of Parseval’s formula gives for \( m = 0, 1, \ldots \)

\[
2\pi \int_0^\infty f(t) t^m e^{-t} \, dt = m! \int_{-\infty}^\infty \hat{f}(x) (1 - ix)^{-m-1} \, dx.
\]

The integral on the right is bounded by \( (\int_{|x| \geq 3} |x|^{-m-1} \, dx) \cdot \| \hat{f} \|_\infty = O(3^{-m}) \), hence
Now Theorem A implies $f \equiv 0$. Q.E.D.

Let $K, K'$ be closed cones in $\mathbb{R}^n$; we say $K'$ is strongly enclosed by $K$ if $\{ x \in K' : |x| = 1 \}$ is in the interior of $K$. We now state our first main result:

**Theorem B'.** Let $f$ be a tempered function on $\mathbb{R}^n$ having a spectral gap containing 0, and nonnegative a.e. on the closed convex cone $K$. Let $K'$ be any closed cone strongly enclosed by $K$. Then, for some $\delta = \delta(f ; K') > 0$,

$$\int_K f(t) e^{\delta |t|} \, dt < \infty.$$

**Corollary.** In the hypotheses of Theorem B', if the cone complementary to $K$ is minor, then $f$ vanishes identically.

The proof of Theorem B' is in principle like that sketched for Theorem B, but complicated technically. The Corollary then follows using Theorem A'.

**Remark.** The hypothesis of nonnegativity on $K$ can be weakened to having range in a sector of opening less than $\pi$.

3. Logan (loc. cit.) also established a relation between a spectral gap about 0 and exponential decay of the Poisson integral [5, Theorems 6.2.3 and 6.3.1]:

**Theorem C.** Let $f \in L^\infty(\mathbb{R})$. The spectrum of $f$ is disjoint from $(-a, a)$ if and only if the Poisson integral

$$u(x ; y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \left( \frac{y}{(x - \xi)^2 + y^2} \right) d\xi$$

of $f$ satisfies

$$|u(x ; y)| \leq A e^{-\alpha y}, \quad y > 0,$$

where $A$ is a positive constant independent of $x$.

This theorem readily implies that of Paley and Wiener. Logan’s proof uses analytic functions. I gave [8, p. 152] a proof using only Fourier analysis, which can be extended to $n$ dimensions. Letting $B(x^0 ; a)$ denote the open ball in $\mathbb{R}^n$ with center $x^0$ and radius $a$, we have our second main result:

**Theorem C'.** Let $f$ be a locally integrable function on $\mathbb{R}^n$ such that

$$\int (1 + |x|^2)^{-(n+1)/2} |f(x)| \, dx < \infty.$$

The spectrum of $f$ is disjoint from $B(0 ; a)$ if and only if the Poisson integral
of $f$ satisfies, for every $\varepsilon > 0$,

\[(4) \int |u(x, y)|(1 + |x|^2)^{-(\alpha + 1)/2} \, dx \leq A(\varepsilon)e^{-(\alpha - \varepsilon)y}, \quad y > 0,\]

where $A(\varepsilon)$ is a positive number independent of $x$. If (3) is replaced by the stronger condition $f \in L^\infty$, (4) is to be replaced by

\[(4a) \int |u(x, y)|^4(\varepsilon)^{-\alpha y}, \quad y > 0,\]

The proof requires estimates for “minimal extrapolations” from the interior, as well as the exterior, of a ball; these will be given elsewhere.

As in the case $n = 1$, the “only if” part of the theorem can be strengthened when $f \in L^\infty$.

If, for $k \in L^1(\mathbb{R}^n)$, we denote by $k_{(y)}$ the “dilated function”:

\[k_{(y)}(x) = y^{-\alpha}k(y^{-1}x); \quad x \in \mathbb{R}^n, y > 0,\]

then Theorem C’ (in the case $f \in L^\infty$) may be written: Let

\[(5) \quad k(t) = c \varepsilon^{-\alpha \varepsilon}t^{-(\alpha + 1)/2} \quad (so \ that \ \hat{k}(\varepsilon) = e^{-(\alpha - \varepsilon)\varepsilon}); \ the \ condition\]

\[(6) \quad |(f \ast k_{(y)})(x)| \leq A(\varepsilon)\hat{k}((\alpha - \varepsilon)\varepsilon), \quad y > 0,\]

holds for all $\varepsilon > 0$, if and only if the spectrum of $f$ is disjoint from $B(0; a)$.

Now, this proposition can be established for a large class of kernels $k(t)$ in place of (5), using exactly the same method; in particular, for $k(t) = e^{-|t|^2}$, a result obtained otherwise by Ehrenpreis and Malliavin; see [3, Corollary 5]. With this special choice of $k$, we may permit $f$ in (6) to be any tempered distribution.

4. Assuming (4a) holds for a single value of $x$, we can nonetheless obtain spectral information about $f$. First, some notation: a locally integrable function on $\mathbb{R}^n$ is anti-radial if its integral over $B(0; r)$ vanishes for every $r > 0$. Every locally integrable function admits an essentially unique decomposition into a radial and an anti-radial part (for $n = 1$, this is just the even–odd decomposition). We now state our third main result:

**Theorem D.** Let $f$ be a bounded measurable function on $\mathbb{R}^n$ whose Poisson integral $u(x; y)$ satisfies $|u(0; y)| \leq Ae^{-\alpha y}$. Then, the radial part of $f$ has spectrum disjoint from $B(0; a)$.

Combining Theorems C’ and D we deduce a proposition solely about harmonic functions: If $u$ is the Poisson integral of a radial function in
$L^\infty(\mathbb{R}^n)$, (4a) holds for every $x \in \mathbb{R}^n$ if it holds for $x = 0$. For $n = 1$ this is a consequence of a classical Phragmén-Lindelöf theorem, but for $n > 1$ it appears to be new. Another corollary of Theorem D is: If $u$ is a bounded harmonic function on $\mathbb{R}^n \times \mathbb{R}^+$ satisfying $u(0; y) = O(e^{-ay})$ as $y \to + \infty$, for every $a > 0$, then $u(0, y)$ vanishes identically (or, what is the same thing, $u(x; 0 + )$ is an anti-radial function on $\mathbb{R}^n$).

Theorem D also yields a particularly simple proof of the existence of "lacunae" for the wave equation (cf. [3, p. 417] for terminology). Also, the analog of Theorem D for the kernel $k(t) = e^{-|t|^2}$ is valid; this is a refinement of a theorem in [3].

**Proof of Theorem D.** Assume first $f \in L^1(\mathbb{R}^n)$. We may assume $f$ radial, since the anti-radial part contributes nothing to $u(0; y)$. By assumption,

$$\int_{\mathbb{R}^n} f(x) \cdot y(|x|^2 + y^2)^{-(n+1)/2} \, dx \leq Ce^{-ay}.$$ 

Substituting here

$$y(|x|^2 + y^2)^{-(n+1)/2} = A_n \int e^{-y|t|} e^{-i(t, x)} \, dt$$

(where $A_n$ depends only on $n$), and applying Fubini's Theorem, yields (integrations are over $\mathbb{R}^n$):

$$A_n \left| \int \hat{f}(t) \cdot e^{-y|t|} \, dt \right| \leq Ce^{-ay}.$$ 

Writing $\hat{f}(t) = \phi(|t|)$, we have

$$\int_0^{\infty} s^{n-1} \phi(s)e^{-ys} \, ds = O(e^{-ay}), \quad y \to + \infty.$$ 

Now a simple argument shows that $\phi(s)$ must vanish for $s < a$, hence $\hat{f}(t) = 0$ for $|t| < a$, as we wished to show.

The general case, when $f$ need not belong to $L^1$, leads to serious complications; to be able to perform the crucial "Fubini" step, we replace the Fourier kernel $e^{-i(t, x)}$ by that of Bochner [2, p. 112, (5)], in an $n$-dimensional version, and then suitably extend the spectral analysis of Pollard [7]. (This is also applicable to $f$ which satisfy only (3).)

5. Let $G$ denote any l.c.a. group, $\hat{G}$ its dual. Let $E, \hat{E}$ be closed subsets of $G, \hat{G}$ respectively. We say the pair $(E, \hat{E})$ is interpolatory if, for every $f \in L^1(G)$, there exists $f_0 \in L^1(G)$ supported in $E$ such that $f_0(x) = \hat{f}(x)$, $x \in \hat{E}$. This is equivalent to saying that every function in $L^1(\hat{G}; E)$ extends to an element of $L^1(G)$ whose Fourier transform vanishes on $\hat{E}$. For instance, one can show when $G = \mathbb{R}^n$, that this is the case if $\mathbb{R}^n; E$ and $\hat{E}$ are compact. On the other hand, Theorem B' implies that $(E, \hat{E})$ is not inter-
polatory if $\mathbb{R}^n \setminus E$ contains a nonvoid open cone and $\hat{E}$ has interior. Thus, the Fourier transforms of functions supported on proper subcones of $\mathbb{R}^n$ are constrained in their local behavior, they possess "local structure". The detailed nature of this local structure was somewhat clarified in [9] for the analogous situation in $L^\infty(\mathbb{R}^n)$. For $n = 1$ one can show that the presence of arbitrarily long intervals in the complement of the spectrum already forces local structure.

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