

## DOUBLE CENTRALIZERS OF PEDERSEN'S IDEAL OF A $C^*$ -ALGEBRA. II

BY A. J. LAZAR AND D. C. TAYLOR

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**1. Introduction.** This note is a sequel of [1], to which we refer for motivation and basic terminology. In this note we develop a comprehensive spectral theory and functional calculus for the double centralizers of Pedersen's ideal. The full details of our discussion will appear elsewhere, and we intend to pursue related topics in subsequent papers.

From now on  $A$  will denote a  $C^*$ -algebra,  $K_A$  its Pedersen ideal (or simply  $K$  if  $A$  is understood), and  $\Gamma(K)$  the double centralizers of  $K$ . As in [1] we will view the double centralizers of  $A$ , denoted by  $\Gamma(A)$  or  $M(A)$ , and  $A$  as subalgebras of  $\Gamma(K)$ .

**2. Spectral theory and a functional calculus.** For each  $a \in K$  let  $\mathcal{L}_a$  and  $\mathcal{R}_{a^*}$  denote the closed left and right ideals of  $A$  generated by  $a$  and  $a^*$  respectively. Note that  $\mathcal{L}_a$  and  $\mathcal{R}_{a^*}$  are subsets of  $K$  (see [1]). Now let  $\Gamma_a$  denote the set of all pairs  $(U, V)$  that satisfy the following: (i)  $U$  and  $V$  are bounded linear operators on  $\mathcal{L}_a$  and  $\mathcal{R}_{a^*}$  respectively; (ii)  $xU(y) = V(x)y$  for each  $x \in \mathcal{R}_{a^*}$  and  $y \in \mathcal{L}_a$ .

Let  $(S, T)$  and  $(U, V)$  belong to  $\Gamma_a$  and let  $\alpha$  be a complex number. Then it is clear that  $(S + U, T + V)$ ,  $(\alpha U, \alpha V)$ , and  $(SU, VT)$  belong to  $\Gamma_a$ . Moreover, if we define  $S^*$  on  $\mathcal{R}_{a^*}$  by the formula  $S^*(x) = S(x^*)^*$  and similarly define  $T^*$  on  $\mathcal{L}_a$ , then  $(T^*, S^*)$  belongs to  $\Gamma_a$ . Consequently,  $\Gamma_a$  is a  $*$ -algebra when provided with the following operations: (i)  $(S, T) + (U, V) = (S + U, T + V)$ ; (ii)  $\alpha(S, T) = (\alpha S, \alpha T)$ ; (iii)  $(S, T)(U, V) = (SU, VT)$ ; (iv)  $(S, T)^* = (T^*, S^*)$ .

**PROPOSITION 2.1.** *If  $(S, T) \in \Gamma_a$ , then  $\|S\|^2 = \|S^*\|^2 = \|T\|^2 = \|T^*S\|$ . Consequently, the  $*$ -algebra  $\Gamma_a$  provided with the norm  $\|(S, T)\| = \|S\|$  is a  $C^*$ -algebra with identity.*

For each double centralizer  $(S, T)$  of  $K$  define  $\lambda_a(S) = S|_{\mathcal{L}_a}$ ,  $\rho_a(T) = T|_{\mathcal{R}_{a^*}}$ , and  $\phi_a((S, T)) = (\lambda_a(S), \rho_a(T))$ . Since  $\mathcal{L}_a$  and  $\mathcal{R}_{a^*}$  are invariant under  $S$  and  $T$  respectively,  $\phi_a$  is a well-defined map of  $\Gamma(K)$  into  $\Gamma_a$ . Moreover, it is clear that  $\phi_a$  is a  $*$ -homomorphism.

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**THEOREM 2.2.** *Suppose  $\{e_\lambda\}_{\lambda \in I}$  is a positive approximate identity for  $A$  contained in  $K$ . Then, for each  $x \in \Gamma(K)$ , the following statements are equivalent: (i)  $x$  is regular in  $\Gamma(K)$ ; (ii)  $\phi_a(x)$  is regular in  $\Gamma_a$  for each  $a \in K$ ; (iii)  $\phi_{e_\lambda}(x)$  is regular in  $\Gamma_{e_\lambda}$  for each  $\lambda \in I$ .*

**SKETCH OF PROOF.** It is clear that (i) implies (ii) and (ii) implies (iii), so assume  $(S, T)$  is a double centralizer of  $K$  for which (iii) holds. For the moment assume that as maps  $S^{-1}$  and  $T^{-1}$  exist. It is straightforward to show that  $(S^{-1}, T^{-1})$  is a double centralizer of  $K$ . Therefore,  $(S, T)$  is regular in  $\Gamma(K)$  and (i) holds. It follows that we need to show  $S$  and  $T$  are one-to-one and onto maps.

The fact that  $S$  and  $T$  are one-to-one maps follows directly from (iii). To prove that  $S$  and  $T$  are onto, it will suffice to show that  $K^+ \subset S(K)$ . Let  $a \in K^+$ . Since  $K^+$  is the smallest invariant face of  $A^+$  containing  $\{e_\lambda\}_{\lambda \in I}$ , there exist elements  $\{e_{\lambda_i}\}_{i=1}^n$  chosen from our approximate identity, unitary elements  $\{u_i\}_{i=1}^n$  in  $\bar{A}$ , and positive scalars  $\{\alpha_i\}_{i=1}^n$  such that  $0 \leq a \leq \sum_{i=1}^n \alpha_i u_i^* e_{\lambda_i} u_i$ . By [3, Corollary 1.2, p. 73], there are elements  $\{z_i\}_{i=1}^n$  in  $A$  such that  $a = \sum_{i=1}^n z_i^* z_i$  and  $z_i z_i^* \leq \alpha_i e_{\lambda_i}$ ,  $1 \leq i \leq n$ . But, by [2, Lemma 1.1, p. 132],  $z_i z_i^* \leq \alpha_i e_{\lambda_i}$  implies  $z_i^* \in \mathcal{L}_{e_{\lambda_i}}$ . Since  $\lambda_{e_{\lambda_i}}(S)$  maps  $\mathcal{L}_{e_{\lambda_i}}$  onto itself, we have, for each  $z_i^*$ , a  $y_i \in \mathcal{L}_{e_{\lambda_i}}$  such that  $\lambda_{e_{\lambda_i}}(S)(y_i) = S(y_i) = z_i^*$ . But  $a = \sum_{i=1}^n z_i^* z_i = \sum_{i=1}^n S(y_i) z_i = S(\sum_{i=1}^n y_i z_i)$  and our proof is complete.

Let  $\mathcal{B}$  be an algebra with identity  $e$  and let  $x \in \mathcal{B}$ . The spectrum of  $x$  with respect to  $\mathcal{B}$ , denoted by  $\sigma_{\mathcal{B}}(x)$ , is defined to be the set of all complex numbers  $\lambda$  for which  $\lambda e - x$  is singular in  $\mathcal{B}$ .

**COROLLARY 2.3.** *If  $x \in \Gamma(K)$ , then*

$$\sigma_{\Gamma(K)}(x) = \bigcup_{a \in K} \sigma_{\Gamma_a}(\phi_a(x)) = \bigcup_{\lambda \in I} \sigma_{\Gamma_{e_\lambda}}(\phi_{e_\lambda}(x)).$$

**COROLLARY 2.4.** *For each  $x \in \Gamma(K)$ ,  $\sigma_{\Gamma(K)}(x)$  is nonempty.*

**COROLLARY 2.5.** *If  $A$  has a countable approximate identity, then, for each  $x \in \Gamma(K)$ ,  $\sigma_{\Gamma(K)}(x)$  can be expressed as a countable union of compact sets.*

**COROLLARY 2.6.** *If  $x$  is a hermitian element of  $\Gamma(K)$ , then  $\sigma_{\Gamma(K)}(x)$  is real.*

**COROLLARY 2.7.** *If  $\mathcal{A}$  is a maximal selfadjoint commutative subalgebra of  $\Gamma(K)$  and  $x \in \mathcal{A}$ , then  $\sigma_{\mathcal{A}}(x) = \sigma_{\Gamma(K)}(x)$ .*

Let  $(S, T)$  be a normal double centralizer of  $K$  and let  $f$  be a complex-valued continuous function defined on  $\sigma_{\Gamma(K)}((S, T))$ . By virtue of Corollary 2.3,  $f$  is continuous on each set  $\sigma_{\Gamma_a}(\phi_a(S, T))$ . Since  $\phi_a$  is a  $*$ -homomorphism,  $\phi_a(S, T)$  is a normal element of  $\Gamma_a$ . Therefore,  $f(\phi_a(S, T))$  can be defined in  $\Gamma_a$  in the usual way. For each  $a \in K$ , let  $(f(\lambda_a(S)), f(\rho_a(T)))$  be the pair

given by  $f(\phi_a(S, T))$ . Now define the linear maps  $f(S):K \rightarrow K$  and  $f(T):K \rightarrow K$  by the formulas  $f(S)(a) = f(\lambda_a(S))(a)$  and  $f(T)(a) = f(\rho_a(T))(a)$  for each  $a \in K$ . Denote the pair  $(f(S), f(T))$  by  $f((S, T))$ .

**LEMMA 2.8.** *The pair  $f((S, T))$  is a double centralizer of  $K$  and, for each  $a \in K$ ,  $\phi_a(f((S, T))) = f(\phi_a((S, T)))$ .*

**SKETCH OF PROOF.** Let  $a, b \in K$ . By virtue of the Stone-Weierstrass theorem, there exists a sequence  $\{f_n\}$  of complex-valued continuous functions on  $\sigma_{\Gamma_a}(\phi_a(S, T)) \cup \sigma_{\Gamma_b}(\phi_b(S, T))$  such that each function  $f_n$  is of the form

$$f_n(v) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} v^i \bar{v}^j$$

and

$$f_n \rightarrow f|_{\sigma_{\Gamma_a}(\phi_a(S, T)) \cup \sigma_{\Gamma_b}(\phi_b(S, T))}$$

uniformly. Hence

$$af(S)(b) = \lim af_n(\lambda_b(S))(b) = \lim af_n(S)(b) = \lim f_n(T)(a)b = f(T)(a)b.$$

Therefore,  $f((S, T))$  is a double centralizer of  $K$ . The remainder of the proof follows directly.

**THEOREM 2.9 (SPECTRAL MAPPING THEOREM).** *Let  $x$  be a normal element in  $\Gamma(K)$ . If  $f$  is a complex-valued continuous function on  $\sigma_{\Gamma(K)}(x)$ , then  $\sigma_{\Gamma(K)}(f(x)) = f(\sigma_{\Gamma(K)}(x))$ .*

**PROOF.** By virtue of Theorem 2.2 and Lemma 2.8,

$$\begin{aligned} \sigma_{\Gamma(K)}(f(x)) &= \bigcup_{a \in K} \sigma_{\Gamma_a}(\phi_a(f(x))) = \bigcup_{a \in K} \sigma_{\Gamma_a}(f(\phi_a(x))) \\ &= \bigcup_{a \in K} f(\sigma_{\Gamma_a}(\phi_a(x))) = f(\bigcup_{a \in K} \sigma_{\Gamma_a}(\phi_a(x))) \\ &= f(\sigma_{\Gamma(K)}(x)). \end{aligned}$$

Let  $x$  be a normal element in  $M(A)$  and  $f$  a complex-valued continuous function on  $\sigma_{M(A)}(x)$ . Put  $f|_{\sigma_{\Gamma(K)}(x)} = \bar{f}$ . If  $f(x)$  is the element in  $M(A)$  given by the usual spectral theorem and  $\bar{f}(x)$  is the element in  $\Gamma(K)$  given by Lemma 2.8, then it is straightforward to show  $f(x) = \bar{f}(x)$ .

**THEOREM 2.10.** *Let  $x$  be a normal element in  $\Gamma(K)$ ,  $f$  and  $g$  complex-valued continuous functions on  $\sigma_{\Gamma(K)}(x)$ ,  $h$  a complex-valued continuous function on  $\sigma_{\Gamma(K)}(g(x))$ , and  $\alpha, \beta$  complex numbers. Then the following statements hold:*

- (i)  $\alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x)$ ;
- (ii)  $f(x)g(x) = f \cdot g(x)$ ;
- (iii) if  $f$  has a power series expansion  $f(v) = \sum_{k=0}^{\infty} \alpha_k v^k$  valid in  $\sigma_{\Gamma(K)}(x)$ , then  $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ ;
- (iv) if  $x$  is hermitian and  $f$  is real-valued, then  $f(x)$  is hermitian;
- (v)  $F(x) = h(g(x))$ , where  $F(v) = h(g(v))$ .

**COROLLARY 2.11.** *Let  $x$  be a normal element in  $\Gamma(K)$ . Then the map  $f \rightarrow f(x)$  of  $C(\sigma_{\Gamma(K)}(x))$  into  $\Gamma(K)$  is a  $*$ -isomorphism which is continuous under the compact-open and  $\kappa$ -topologies.*

**REMARK.** The map  $f \rightarrow f(x)$  is not in general bicontinuous even if  $A$  is commutative.

**COROLLARY 2.12.** *If  $x$  is a hermitian element in  $\Gamma(K)$ , then the following statements are equivalent :*

- (i)  $\sigma_{\Gamma(K)}(x) \geq 0$ ;
- (ii)  $x$  is of the form  $yy^*$ , where  $y \in \Gamma(K)$ ;
- (iii)  $x$  is of the form  $h^2$ , where  $h$  is a hermitian element in  $\Gamma(K)$ .

We say that an element  $x$  in  $\Gamma(K)$  is positive, denoted by  $x \geq 0$ , if it is hermitian and satisfies any of the equivalent conditions of Corollary 2.12. We let  $\Gamma(K)^+$  denote the set of all positive elements of  $\Gamma(K)$ .

**COROLLARY 2.13.** *The algebra  $\Gamma(K)$  is the linear span of its positive elements.*

Now let us view  $A$  as a  $C^*$ -algebra of operators on the Hilbert space  $H$  such that  $A[H] \equiv \{x(h)|x \in A, h \in H\}$  is dense in  $H$ , or equivalently,  $A[H] = H$ . The algebra  $\Gamma(K)$  can be viewed as the  $*$ -algebra of all operators  $T$  acting in the dense subspace  $K[H]$  which have the property  $xT + Ty$  is bounded on  $K[H]$  for each  $x, y \in K$  and its unique extension to  $H$  belongs to  $K$ .

**COROLLARY 2.14.** *Let  $x \in \Gamma(K)$ . Then  $x \geq 0$  if and only if  $(x(h), h) \geq 0$  for each  $h \in K[H]$ .*

**COROLLARY 2.15.** *The set  $\Gamma(K)^+$  is a convex cone and  $\Gamma(K)^+ \cap (-\Gamma(K)^+) = \{0\}$ .*

**COROLLARY 2.16.** *The algebra  $\Gamma(K)$  is symmetric.*

The relation  $x - y \geq 0$  is an order relation on  $\Gamma(K)$  compatible with the real vector space structure of  $\Gamma(K)$ ; we denote this relation by  $x \geq y$  or  $y \leq x$ .

**PROPOSITION 2.17.** *Let  $a, b, x$  be elements of  $\Gamma(K)$ . If  $a \leq b$ , then  $x^*ax \leq x^*bx$ . Moreover, if  $a$  and  $b$  are regular and  $b \geq a \geq 0$ , then  $a^{-1} \geq b^{-1}$ .*

**PROPOSITION 2.18.** *A normal element  $x$  in  $\Gamma(K)$  belongs to  $M(A)$  if and only if  $\sigma_{\Gamma(K)}(x)$  is bounded.*

**EXAMPLE.** If  $x$  is not normal it may happen that  $\sigma_{\Gamma(K)}(x)$  is bounded yet  $x \notin M(A)$ . Suppose  $A$  is the algebra of all sequences of  $2 \times 2$  matrices which converge to the zero matrix. The Pedersen ideal  $K$  of  $A$  is the space

of all sequences that vanish except finitely many times,  $M(A)$  is the space of all bounded sequences, and  $\Gamma(K)$  is the space of all sequences. Let  $T$  be a nilpotent matrix of norm 1. Then  $x = \{nT\}_{n=1}^\infty$  belongs to  $\Gamma(K)$ , but not to  $M(A)$ . However,  $\sigma_{\Gamma(K)}(x) = \{0\}$ .

**PROPOSITION 2.19.** *If  $x$  is a normal element in  $M(A)$ , then the closure of  $\sigma_{\Gamma(K)}(x)$  is  $\sigma_{M(A)}(x)$ .*

**PROPOSITION 2.20.** *If  $x$  is a normal element in  $A$ , then  $\sigma_{M(A)}(x)/\{0\} \subset \sigma_{\Gamma(K)}(x) \subset \sigma_{M(A)}(x)$ .*

Up to now the functional calculus that we have developed has been for normal elements of  $\Gamma(K)$ . We will now develop a functional calculus for nonnormal elements.

Let  $(S, T)$  be a double centralizer of  $K$  and  $\Delta$  an open subset of the complex plane. For each  $a \in K$ , the set  $\sigma_{B(\mathcal{L}_a)}(\lambda_a(S)) \cup \sigma_{B(\mathcal{R}_{a^*})}(\rho_a(T))$  is compact and contained in  $\sigma_{\Gamma(K)}((S, T))$ . Consequently, we can find a bounded open set  $\Omega$  with the following properties:

$$\sigma_{B(\mathcal{L}_a)}(\lambda_a(S)) \cup \sigma_{B(\mathcal{R}_{a^*})}(\rho_a(T)) \subseteq \Omega \subseteq \Delta;$$

$\Omega$  has a finite number of components  $\Omega_\mu$ ; each  $\Omega_\mu$  is bounded by a finite number of simple closed rectifiable curves  $B_{\mu\nu}$ ;  $\Omega$  has positive distance from the boundary of  $\Delta$ . We assign positive orientation to each  $B_{\mu\nu}$  in the usual manner and let  $B = \bigcup B_{\mu\nu}$ . We call  $B$  an oriented envelope of  $\sigma_{B(\mathcal{L}_a)}(\lambda_a(S)) \cup \sigma_{B(\mathcal{R}_{a^*})}(\rho_a(T))$  with respect to  $\Delta$ . Now let  $f$  be a complex-valued analytic function in  $\Delta$ . Define

$$f(\lambda_a(S)) = \frac{1}{2\pi i} \int_B f(z)(z\lambda_a(I) - \lambda_a(S))^{-1} dz$$

and

$$f(\rho_a(T)) = \frac{1}{2\pi i} \int_B f(z)(z\rho_a(I) - \rho_a(T))^{-1} dz,$$

where the integrals are in the usual Riemann-Stieltjes sense. By virtue of Cauchy's theorem,  $f(\lambda_a(S))$  and  $f(\rho_a(T))$  depend upon  $f$  and not on the domain  $\Delta$ . Now define the linear maps  $f(S): K \rightarrow K$  and  $f(T): K \rightarrow K$  by the formulas  $f(S)(a) = f(\lambda_a(S))(a)$  and  $f(T)(a) = f(\rho_a(T))(a)$  for each  $a \in K$ . Denote the pair  $(f(S), f(T))$  by  $f((S, T))$ .

**THEOREM 2.21.** *The pair  $(f(S), f(T))$  is a double centralizer of  $K$  and  $(\lambda_a f(S), \rho_a f(T)) = (f(\lambda_a(S)), f(\rho_a(T)))$  for each  $a \in K$ . Consequently,  $\sigma_{\Gamma(K)}(f((S, T))) = f(\sigma_{\Gamma(K)}((S, T)))$ .*

For each  $x \in \Gamma(K)$  let  $\mathcal{F}(x)$  denote the family of all complex-valued

functions  $f$  which are analytic on some neighborhood of  $\sigma_{\Gamma(K)}(x)$ . (The neighborhood need not be connected and can depend on  $f \in \mathcal{F}(x)$ .)

**THEOREM 2.22.** *If  $f$  and  $g$  are in  $\mathcal{F}(x)$ ,  $h$  is in  $\mathcal{F}(g(x))$ , and  $\alpha, \beta$  are complex numbers, then the following statements hold:*

- (i)  $\alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x)$ ;
- (ii)  $f(x)g(x) = f \cdot g(x)$ ;
- (iii) *if  $f$  has a power series expansion  $f(v) = \sum_{k=0}^{\infty} \alpha_k v^k$  valid in  $\sigma_{\Gamma(K)}(x)$ , then  $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ ;*
- (iv) *if  $\bar{f}(v) = \overline{f(\bar{v})}$ , then  $\bar{f} \in \mathcal{F}(x^*)$  and  $\bar{f}(x^*) = f(x)^*$ ;*
- (v)  $F(x) = h(g(x))$ , where  $F(v) = h(g(v))$ .

**PROPOSITION 2.23.** *Let  $\{f_\lambda\}$  be in  $\mathcal{F}(x)$  and suppose that all the functions  $f_\lambda$  are analytic in some fixed neighborhood  $V$  of  $\sigma_{\Gamma(K)}(x)$ . Then, if  $f_\lambda$  converges uniformly on compact subsets of  $V$  to the function  $f$ ,  $f_\lambda(x)$  converges to  $f(x)$  in the  $\kappa$ -topology.*

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DEPARTMENT OF MATHEMATICS, TEL-AVIV UNIVERSITY, TEL-AVIV, ISRAEL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65201