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A FUNDAMENTAL SOLUTION FOR A SUBELLIPTIC OPERATOR[1]

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1. **Introduction.** Let $\mathscr{L}: C^\infty(M) \to C^\infty(M)$ be a formally selfadjoint differential operator of order 2 on the Riemannian manifold M. $\mathscr{L}$ is said to be *subelliptic of order* ε $(0 < \varepsilon < 1)$ at $x \in M$ if there exist a neighborhood V of x and a constant $c > 0$ such that for all $u \in C_0^\infty(V)$,

$$\|u\|_\varepsilon^2 \leqq c(|(\mathscr{L}u, u)| + \|u\|^2), \tag{1}$$

where $\|u\|$ is the L^2 norm and $\|u\|_\varepsilon$ is the Sobolev norm of order ε. According to a fundamental theorem of Kohn and Nirenberg [**3**], subelliptic operators are hypoelliptic and satisfy the *a priori* estimates

$$\|u\|_{s+2\varepsilon}^2 \leqq c_s(\|\mathscr{L}u\|_s^2 + \|u\|^2), \qquad u \in C_0^\infty(V), \tag{2}$$

for each $s \geqq 0$.

In this note we shall display an operator on a Euclidean space which is subelliptic of order $\frac{1}{2}$ at each point and construct an explicit integral operator which inverts it.

2. **Construction of the operator.** Let N be the nilpotent Lie group whose underlying manifold is $\boldsymbol{C}^n \times \boldsymbol{R}$ with coordinates $(z_1, \ldots, z_n, t) = (z, t)$ and whose group law is

$$(z, t)(z', t') = (z + z', t + t' + 2\operatorname{Im} z \cdot z')$$

where $z \cdot z' = \sum_1^n z_j \bar{z}'_j$. Letting $z = x + iy$, then, $x_1, \ldots, x_n, y_1, \ldots, y_n, t$ are real coordinates on N. We set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

$$Z_j = \tfrac{1}{2}(X_j - iY_j), \qquad \bar{Z}_j = \tfrac{1}{2}(X_j + iY_j).$$

The following proposition is easily verified.

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LEMMA 1. *$X_1, \dots, X_n, Y_1, \dots, Y_n, T$ are a basis for the Lie algebra of N.*

We impose the left-invariant metric on N which makes this basis orthonormal at each point and note that the induced volume element is Lebesgue measure, which we denote by $d(z, t)$.

THEOREM 1. *The operator*

$$\mathscr{L} = \sum_1^n \left[-\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - |z_j|^2 \frac{\partial^2}{\partial t^2} + i\frac{\partial}{\partial t}\left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \right]$$

is left-invariant and is subelliptic of order $\frac{1}{2}$ at each $x \in N$.

PROOF. One easily sees that $\mathscr{L} = -\frac{1}{2}\sum_1^n (Z_j\bar{Z}_j + \bar{Z}_jZ_j)$, which by Lemma 1 implies left-invariance. Moreover, since Z_j is the formal adjoint of $-\bar{Z}_j$, we have

$$(3) \qquad (\mathscr{L}u, u) = \frac{1}{2}\sum_1^n (\|\bar{Z}_ju\|^2 + \|Z_ju\|^2), \qquad u \in C_0^\infty(N).$$

We invoke the following special case of a theorem of Kohn [**2**] and Radkevič [**5**]:

LEMMA 2. *Let V be a compact set in a Riemannian manifold M, and let $L_1, \dots, L_N$ be complex vector fields on M whose linear span is closed under complex conjugation and such that $\{L_j\}_1^N \cup \{[L_j, L_k]\}_{j,k=1}^N$ spans the tangent space at each $x \in V$. Then there exists $c > 0$ such that for all $u \in C_0^\infty(V)$,*

$$\|u\|_{1/2}^2 \leqq c\left(\sum_1^N \|L_ju\|^2 + \|u\|^2\right).$$

The hypotheses of Lemma 2 are satisfied if we take the L_j's to be $Z_1, \dots, Z_n, \bar{Z}_1, \dots, \bar{Z}_n$, since $[\bar{Z}_j, Z_j] = 2iT$. Hence (3) implies (1), and the theorem is proved.

REMARK. N is the nilpotent part in the Iwasawa decomposition of the holomorphic automorphism group of the Siegel domain

$$\left\{\zeta \in \boldsymbol{C}^{n+1} : \sum_1^n |\zeta_j|^2 - \operatorname{Im}\zeta_{n+1} < 0\right\},$$

and it may be identified with the boundary of the domain via the correspondence $(z, t) \leftrightarrow (z_1, \dots, z_n, t + i\sum_1^n |z_j|^2)$. Under this identification, $-2\sum_1^n Z_j\bar{Z}_j$ is just the "tangential complex Laplacian" $\Box_b$ of J. J. Kohn (cf. [**1**]), and hence $\mathscr{L} = \frac{1}{4}(\Box_b + \bar{\Box}_b)$. Also, note that when $n = 1$, the operator $\bar{Z} = (\partial/\partial\bar{z}) - iz(\partial/\partial t)$ is the "unsolvable" operator of H. Lewy [**4**].

3. **Construction of the fundamental solution.** Following Stein [**6**], we

introduce the group $\{\delta_r : 0 < r < \infty\}$ of *dilations* on N defined by $\delta_r(z, t) = (rz, r^2 t)$, which satisfy the distributive law $\delta_r((z, t)(z', t')) = (\delta_r(z, t))(\delta_r(z', t'))$, and we define the *norm* function $\rho(z, t) = (|z|^4 + t^2)^{1/4}$ (where $|z|^2 = z \cdot z$), which satisfies $\rho(\delta_r(z, t)) = r\rho(z, t)$. By analogy with the fact that $|x|^{2-m}$ is (a constant multiple of) the fundamental solution of the Laplacian on $\boldsymbol{R}^m$ with source at 0, we now prove

THEOREM 2. *$c_n \rho^{-2n}$ is a fundamental solution for $\mathscr{L}$ with source at* 0, *where*

$$c_n = \left[n(n + 2) \int_N |z|^2 (\rho(z, t)^4 + 1)^{-(n+4)/2} \, d(z, t) \right]^{-1}.$$

In other words, for any $u \in C_0^\infty(N)$, $(\mathscr{L}u, c_n \rho^{-2n}) = u(0)$.

PROOF. Given $\varepsilon > 0$, let $\rho_\varepsilon = (\rho^4 + \varepsilon^4)^{1/4}$; a simple calculation then shows that

$$(\mathscr{L}\rho_\varepsilon^{-2n})(z, t) = \varepsilon^{-2n-2} \phi(\delta_{1/\varepsilon}(z, t))$$

where

$$\phi(z, t) = n(n + 2)|z|^2 (\rho(z, t)^4 + 1)^{-(n+4)/2}.$$

From the fact that $\varepsilon^{-2n-2} \int_N \phi \circ \delta_{1/\varepsilon} = \int_N \phi = c_n^{-1} < \infty$ and the fact that $\delta_{1/\varepsilon}(V) \to N$ as $\varepsilon \to 0$ for any neighborhood V of 0, it now follows easily that for any $u \in C_0^\infty(N)$,

$$(\mathscr{L}u, c_n \rho^{-2n}) = \lim_{\varepsilon \to 0} (\mathscr{L}u, c_n \rho_\varepsilon^{-2n}) = \lim_{\varepsilon \to 0} (u, c_n \mathscr{L} \rho_\varepsilon^{-2n}) = u(0),$$

and the theorem is proved.

Since $\mathscr{L}$ is left-invariant, we deduce immediately

COROLLARY 1. *If $f \in C_0^\infty(N)$, then the function $u = f * (c_n \rho^{-2n})$ is a solution of $\mathscr{L}u = f$, where $*$ denotes convolution on the group N.*

The hypothesis on f can be relaxed considerably, of course. For example, the convolution integral will converge absolutely provided that $f \in L^{n+1-\varepsilon} \cap L^{n+1+\varepsilon}$ for some $\varepsilon > 0$.

4. **Applications.** We shall now prove a precise regularity theorem for $\mathscr{L}$ by means of the theory of singular integrals on nilpotent groups (cf. [**6**] and the references given there). A *singular integral kernel* on N is a function of the form $\Omega \rho^{-2n-2}$ where Ω is a smooth function on $N - \{0\}$ satisfying $\Omega(\delta_r(z, t)) = \Omega(z, t)$ for all $r > 0$ and $\int_{a < \rho(z,t) < A} \Omega(z, t) \, d(z, t) = 0$ for all $0 < a < A < \infty$. If ψ is a singular integral kernel, the operator $f \to f * \psi$, the convolution integral being defined in a suitable principal-value sense, enjoys the same basic properties as Calderon-Zygmund operators on $\boldsymbol{R}^m$: it is bounded on L^p, $1 < p < \infty$, and is weak type (1, 1).

Theorem 3. *Let* $u = f * (c_n \rho^{-2n})$ *as in Corollary* 1. *Then the operators taking* f *to* $X_j X_k u$, $Y_j Y_k u$, $X_j Y_k u$, $Y_j X_k u$ $(j, k = 1, \ldots, n)$ *and* Tu (*but not* $X_j Tu$, $Y_j Tu$, *or* $T^2 u$) *are bounded on* L^p, $1 < p < \infty$, *and are weak type* $(1, 1)$.

Proof. Computations similar to those in the proof of Theorem 2 show that the distribution derivatives $T\rho^{-2n}$, $X_j Y_k \rho^{-2n}$, $Y_j X_k \rho^{-2n}$, and, for $j \neq k$, $X_j X_k \rho^{-2n}$ and $Y_j Y_k \rho^{-2n}$ are singular integral kernels, and the distribution derivatives $X_j^2 \rho^{-2n}$ and $Y_j^2 \rho^{-2n}$ are singular integral kernels plus multiples of the Dirac δ-function at 0. The theorem now follows immediately from the definition of u and the left-invariance of X_j, Y_j, and T.

By the same reasoning, of course, we can estimate higher derivatives of u in terms of appropriate derivatives of f by shifting some of the derivatives onto f in the convolution defining u. This yields a very precise interpretation of the estimates (2) as well as their extension to L^p, $p \neq 2$: Passage from f to u gains one derivative in the T direction and two derivatives in all directions orthogonal to T.

We hope to elaborate on these ideas in a future publication.

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