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FIBERINGS OF MANIFOLDS AND TRANSVERSALITY

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1. **Introduction and statement of results.** This paper, intended as a pendant to some earlier work of the author and J. Morgan [**3**] (and also of Quinn [**4**] and Jones [**1**]), deals with the following question: Given a manifold M^n which is homotopically equivalent to the total space of a fibration with a manifold K^k as fiber, when may M^n itself be fibered by K^k? (Note that if $M^n \simeq U$, and U is the total space of a K^k-bundle over a finite complex, then we may as well assume that U is a manifold with boundary and that it fibers over a manifold with boundary.)

We give sufficient conditions that this may be done, assuming that the fiber K^k is 3-connected, M^n is 4-connected, and $n - k$ is odd. With an additional assumption on the structural group of the bundle, these sufficient conditions are quite obviously necessary, thus making the theorem that much more reasonable.

First we need some definitions.

Let V^n be a PL submanifold of a PL manifold U^{n+r} (with PL normal bundle). Let W be a tubular neighborhood of V, and let T be a triangulation of U as a simplicial complex. We say that T is in general position with respect to V^n iff every simplex σ of T is in general position with respect to V^n, i.e., meets V^n in a manifold with boundary of codimension r in σ. Say that T is in general position with respect to V^n, respecting the tube W, if T is in general position with respect to V^n and for each simplex σ of $T, \sigma \cap W$ is a tubular neighborhood of $\sigma \cap V^n$.

If U is a manifold with boundary, define a *shrinking* of U as a codimension-zero submanifold $U' \subset U$ such that $U = U' \cup \partial U' \times I$ (where $\partial U' \times 0$ is identified with $\partial U'$).

Hereafter, all manifolds, bundles, etc. are to be PL. Since we will be working in high codimension, we will not have to worry about distinctions between PL, $\widetilde{\mathrm{PL}}$, etc.

Let M^n be 4-connected, K^k 3-connected, $n - k$ odd, $\geqq 5$.

THEOREM A. *Sufficient conditions that M^n fibers over a manifold V^{n-k} with fiber K^k are the following*:

(i) *There is an r-dimensional PL-bundle η over M^n so that W^{n+r}, the total space of the disc bundle, fibers over the manifold with boundary Z^{n-k+r} with fiber K^k.*

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(ii) *There are triangulations T of W^{n+r}, S of Z^{n-k+r} and a shrinking Z' of Z such that*

(a) *The projection map π: $W^{n+r} \to Z^{n-k+r}$ is a simplicial map $T \to S$.*

(b) *$W' = \pi^{-1}Z'$ is a tubular neighborhood of M^n and T is in general position with respect to M^n and respecting the tube W'.*

If attention is restricted to K^k-fiberings with structure group a group of simplicial automorphisms of a fixed triangulation of K^k, then these sufficient conditions are necessary.

If we remove the condition that $n - k$ be odd, we obtain the following:

COROLLARY B. *The conditions* (i) *and* (ii) *are sufficient for fibering M^n by K^k provided a certain surgery obstruction in $\pi_{n-k}(G/PL)$ vanishes.*

2. **Sketch of the proof.** First, we leave to the reader the easy part of Theorem A, that is, the necessity of conditions (i) and (ii) when M^n is fibered by K^k so that the structure group is a (discrete) group of automorphisms of a fixed simplicial structure on K^k.

The interesting part is the sufficiency of (i) and (ii). This is also where the ideas of [**3**] are used.

We may assume without loss of generality that r is arbitrarily large. Let Z^{n-k+r}, Z' etc. be as in (i), (ii).

LEMMA 1. *The triangulation S of Z is in Poincaré general position with respect to the Poincaré complex Z'. That is, given any simplex σ of S, $(\sigma \cap Z', \dot{\sigma} \cap Z')$ is a Poincaré pair of codimension r in σ.*

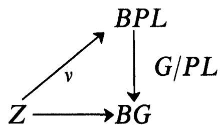
We now recall the chief results [**3**]. Let ν denote the S^{r-1} fibration $\partial Z \subseteq Z$. In [**2**], [**3**], we showed how to construct a certain space $\omega(\nu)$ and map p: $\omega(\nu) \to T(\nu)$ (where $T(\nu)$ is the Thom space). $p: \omega(\nu) \to T(\nu)$ is the "universal example" of a simplexwise Poincaré transverse regular map to $T(\nu)$. We shall not go into details of this construction, but the point is that the construction is natural (with respect to maps of spherical fibrations) and, at least for 4-connected base spaces, finding a section of p is equivalent to finding a PL structure on ν. That is, a fiber homotopy equivalence $\nu \to \xi$, ξ a PL bundle yields a section

$$\omega(\nu) \underset{s}{\overset{p}{\rightleftarrows}} T(\nu)$$

and, conversely, any section arises from such a PL structure on ν. Thus, homotopy classes of sections of p

$$\begin{array}{c} \omega(\nu) \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} \\ T(\ ) \end{array}$$

are in 1–1 correspondence with homotopy classes of liftings



Let $\mathscr{S}(\nu)$ be the set of homotopy classes of sections of $p\colon \omega(\nu) \to T(\nu)$.

LEMMA 2. *Let α be a spherical fibration with a fixed PL structure. Then there is a map*

$$\mathscr{S}(\nu) \xrightarrow{\phi} \mathscr{S}(\nu \oplus \alpha)$$

corresponding to the obvious map PL structures on $\nu \to$ PL structures on $\nu \oplus \alpha$.

The proof of this lemma is easy.

Claim. The fact that Z has a triangulation by S so that S is in Poincaré general position with respect to Z gives rise to a certain section s of p. By Lemma 2, the Spivak normal fibration ν_0 of the $(n-k)$-dimensional Poincaré complex Z has a PL structure. For we set $\alpha =$ PL normal bundle of the manifold with boundary Z. I.e., there is a particular fiber homotopy equivalence $\nu_0 \to \gamma$ for some PL-bundle γ. By surgery theory [**6**], we may find a closed PL manifold V^{n-k} homotopy equivalent to Z. Moreover, with $n-k$ odd, the fact that "surgery obstruction" difficulties vanish means that we produce a sequence of spherical fibration maps $\nu(V^{n-k}) \to \nu_0 \to \gamma$ where $\nu(V^{n-k})$ is the normal bundle of V^{n-k} and the composite is fiber homotopic to a PL bundle equivalence.

It is then possible to identify Z with the disc bundle of a PL bundle $\bar{\nu} \to V^{n-k}$. Moreover, the section $s_0\colon T(\nu) \to \omega(\nu)$ which arises from this PL structure determines the same element in $\mathscr{S}(\nu)$ as the original section s.

At the same time, W is seen to be the r-disc bundle of a PL bundle η_0 over M_0^n, where $M_0^n = \pi^{-1}V^{n-k} \subseteq W$. Our task, then, is to show that the homotopy equivalence

$$M^n \overset{\simeq}{\to} W \underset{\text{proj}}{\overset{\simeq}{\to}} M_0^n$$

has zero normal invariant in $[M, G/\text{PL}]$, for then it follows that M^n is PL equivalent to N_0^n and thus is fibered by K^k.

We detect the triviality of the normal invariant as follows:

The fact that η is a PL bundle over M^n gives rise to a section of p, which we call $t\colon T(\eta) \to \omega(\eta)$. The fact that η has another PL structure, viz. the bundle η_0 over M_0^n gives rise to another section $t_0\colon T(\eta) \to \omega(\eta)$.

It may easily be shown, by the way in which M_0^n was constructed, that there is a commutative diagram

$$(1)\qquad \begin{array}{ccc} \omega(\eta) & \longrightarrow & \omega(\nu) \\ {\scriptstyle p_\eta}\downarrow\ \uparrow{\scriptstyle t_0} & & {\scriptstyle p_\nu}\downarrow\ \uparrow{\scriptstyle s_0} \\ T(\eta) & \longrightarrow & T(\nu) \end{array}$$

Similarly, the assumption that the triangulation T of W is in general position with respect to M^n, respecting the tube W' gives rise to a section $T(\nu) \to \omega(\nu)$ which is t, up to equivalence. We use (ii)(a), to show easily that there is a commutative diagram

$$(2)\qquad \begin{array}{ccc} \omega(\eta) & \longrightarrow & \omega(\nu) \\ {\scriptstyle p_\eta}\downarrow\ \uparrow{\scriptstyle t} & & {\scriptstyle p_\nu}\Big|\ \uparrow{\scriptstyle s} \\ T(\eta) & \longrightarrow & T(\nu) \end{array}$$

Now we use the fact that, for our purposes, the square

$$\begin{array}{ccc} \omega(\eta) & \longrightarrow & \omega(\nu) \\ {\scriptstyle p_\eta}\Big| & & \downarrow{\scriptstyle p_\nu} \\ T(\eta) & \longrightarrow & T(\nu) \end{array}$$

may be regarded as a map of fibrations ([**1**], [**2**], [**3**], [**4**]). In point of strict fact, this may not be true, but the induced map fiber $p_\eta \to$ fiber p_ν induces isomorphism on homotopy groups in dimensions $\geqq 4$, and in view of the fact that M, Z are 4-connected, this is enough.

There is thus a map $\psi : \mathscr{S}(\nu) \to \mathscr{S}(\eta)$ and we have, therefore,

$$[t] = \psi[s] = \psi[s_0] = [t_0] \in \mathscr{S}(\eta).$$

We now apply Lemma 2 again, and this may be visualized in the following form: Let h be the given homotopy equivalence $h: M^n \overset{\simeq}{\to} M_0^n$. Embed the mapping cylinder $\mathscr{M}_h$ in $S^j \times [0, 1]$ (j large) with M^n in $S^j \times 1$, M_0^n in $S^j \times 0$ and the rest of $\mathscr{M}_h$ in $S^j \times (0, 1)$. $(\mathscr{M}_h, M^n \cup M_0^n)$ is a Poincaré pair and, by Lemma 3, we may show that if u is the Spivak normal fibration of the embedding, there is a section $v \in \mathscr{S}(u)$ extending the obvious sections $v_0 \in \mathscr{S}(u|M_0^n)$, $v_1 \in \mathscr{S}(u|M_0^n)$, which exist because M_0^n, M^n are PL manifolds with PL normal bundles. According to [**3**], this guarantees that h has zero normal invariant in $[M, G/\mathrm{PL}]$ and is thus deformable to a PL equivalence.

As to Corollary B, the reader will see that the only problem preventing the proof above from going through in the case $n - k$ even arises from the difficulty in constructing V^{n-k} by actually carrying through the necessary surgery in the middle dimension, i.e., without losing track of the normal bundle.

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