BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, March 1973

DECOMPOSABILITY OF HOMOTOPY LENS SPACES

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Communicated by Glen E. Bredon, September 25, 1972

Let A be the antipodal map of S^p and D^{p+1}. Let f be an equivariant diffeomorphism of $(S^p \times S^p, A \times A)$. Then there is a well-defined free involution $A(f)$ on $\Sigma(f)$, where

$$\Sigma(f) = S^p \times D^{p+1} \cup_f D^{p+1} \times S^p$$

such that $A(f)|S^p \times D^{p+1} = A \times A$ and $A(f)|D^{p+1} \times S^p = A \times A$. In [2] G. R. Livesay and C. B. Thomas have shown that any free involution (Σ^{2p+1}, T) on the homotopy sphere Σ^{2p+1} is decomposable, i.e., there is an equivariant diffeomorphism f of $(S^p \times S^p, A \times A)$ such that (Σ^{2p+1}, T) is equivalent to $(\Sigma(f), A(f))$. For $p =$ odd, let A be a linear Z_n action on S^p and D^{p+1}. We can generalize the notion of decomposable actions to Z_n actions. Using the same argument, they have shown that all free Z_3 actions on homotopy spheres are decomposable. The proof uses the following two well-known facts: (a) $J: KO(RP^p) \to J(RP^p)$ and

$$J: KO(L^{4n-1}(Z_3)) \to J(L^{4n-1}(Z_3))$$

are isomorphisms and (b) $\mathrm{Wh}(Z_2)$ and $\mathrm{Wh}(Z_3)$ are zero. The argument breaks down for Z_n actions, for $n \geqq 4$. Hence they asked if there are similar properties for Z_n actions, for $n \geqq 4$ ([2], [3]). On the other hand, we have studied the analogs for free actions of S^1 and S^3 on homotopy spheres. The same argument works if we replace (a) by some restrictions on the orbit spaces [7].

In this paper we will show that certain free Z_n actions on homotopy spheres are decomposable and the restrictions are nontrivial and necessary.

For $\varepsilon = h$ or s, let $\mathscr{S}^\varepsilon(L^{2m1})$ be the set of (simple, if $\varepsilon = s$) homotopy of complex dimension m. Then $\rho = \sum_{j=1}^m t^{r_j}$ where t is the basic complex one dimensional representation of Z_n defined to be the multiplication by $\exp(2\pi i/n)$. Let $p = [m/2]$, $q = m - [m/2]$. It is clear that

$$S^{2m-1}, S^{2p-1} \times D^{2q}, D^{2p} \times S^{2q-1} \text{ and } S^{2p-1} \times S^{2q-1}$$

are invariant subspaces. Let

AMS(MOS) subject classifications (1970). Primary 57E30.

Key words and phrases. Free Z_n actions, Wall groups, simple homotopy smoothings, decomposable actions, homotopy lens space.

$$L^{2m-1} = L^{2m-1}(n;r_1,\dots,r_m) = S^{2m-1}/\rho,$$
$$M^{2m-1} = M^{2m-1}(n;r_1,\dots,r_m) = S^{2p-1} \times D^{2q}/\rho,$$
$$N^{2m-1} = N^{2m-1}(n;r_1,\dots,r_m) = D^{2p} \times S^{2q-1}/\rho,$$
$$K^{2m-2} = K^{2m-2}(n;r_1,\dots,r_m) = S^{2p-1} \times S^{2q-1}/\rho.$$

For $\varepsilon = h$ or s, let $\mathcal{S}^\varepsilon(L^{2m-1})$ be the set of (simple, if $\varepsilon = s$) homotopy smoothings of L^{2m-1} and G/O be the classifying space for G/O-bundles and let $L_{2m}(Z_n)^0$ be the reduced Wall group of Z_n. By the theorem of T. Petrie [**5**] (see also [**1**]) $L_{2m}(Z_n)^0$ acts freely on $\mathcal{S}^s(L^{2m-1})$. Let

$$\eta : \mathcal{S}^\varepsilon(L^{2m-1}) \to [L^{2m-1}, G/O]$$

be the normal map. W. Browder [**1**] has proved that η is onto if n is odd and according to C. T. C. Wall [**6**] the coker η is Z_2 if n is even.

Let $L^{2p-1} = L^{2p-1}(n;r_1,\dots,r_p)$ and $L^{2q-1} = L^{2q-1}(n;r_{p+1},\dots,r_m)$. Let

$$t_1^\varepsilon : \mathcal{S}^\varepsilon(L^{2m-1}) \xrightarrow{\eta} [L^{2m-1}, G/O] \to [L^{2p-1}, G/O],$$
$$t_2^\varepsilon : \mathcal{S}^\varepsilon(L^{2m-1}) \xrightarrow{\eta} [L^{2m-1}, G/O] \to [L^{2q-1}, G/O],$$
$$\bar{t}_1^\varepsilon : \mathcal{S}^\varepsilon(L^{2m-1}) \xrightarrow{t_1^\varepsilon} [L^{2p-1}, G/O] \to [L^{2p-1}, BO],$$
$$\bar{t}_2^\varepsilon : \mathcal{S}^\varepsilon(L^{2m-1}) \xrightarrow{t_2^\varepsilon} [L^{2q-1}, G/O] \to [L^{2q-1}, BO].$$

PROPOSITION 1. *Let X^{2m-1} be a closed manifold which is homotopy equivalent to $L^{2m-1}(n;r_1,\dots,r_m)$. Suppose that there is an h-cobordism W^{2m-1} of $K^{2m-2}(n;r_1,\dots,r_m)$ to itself so that*

$$X^{2m-1} \cong M^{2m-1}(n;r_1,\dots,r_m) \cup W^{2m-1} \cup N^{2m-1}(n;r_1,\dots,r_m).$$

Then for any homotopy equivalence

$$f : X^{2m-1} \to L^{2m-1}(n;r_1,\dots,r_m),$$

$t_i^h([X^{2m-1}, f]) = 0$ *and* $t_i^s([X^{2m-1}, f]) = 0$ *if f is a simple homotopy equivalence, $i = 1, 2$.*

It is easy to see that for $n \geqq 4$, there is $x \in [L^{2m-1}, G/O]$ such that $x|L^{2p-1} \neq 0$ and if n is even we may choose $x \in \operatorname{Im} \eta$. Suppose $x = \eta([X^{2m-1}, f])$. Hence X^{2m-1} cannot be decomposable. Furthermore, for $w \in L_{2m}(Z_n)^0$,

$$t_1^s(w + [X^{2m-1}, f]) = t_1^s([X^{2m-1}, f]) \neq 0.$$

Since rank $L_{2m}(Z_n)^0 \geqq 1$ for $n \geqq 4$ [**5**], we have

COROLLARY 2. *For $m \geqq 3$ and $n \geqq 4$, there are infinitely many inequiva-*

lent nondecomposable free Z_n actions on homotopy $(2m-1)$-spheres of which the orbit spaces are simple homotopy equivalent to $L^{2m-1}(n;r_1,\ldots,r_m)$.

THEOREM 3. *Let X^{2m-1} be a closed manifold which is homotopy equivalent to $L^{2m-1}(n;r_1,\ldots,r_m)$. Suppose that there is a simple homotopy equivalence*

$$f:X^{2m-1}\to L^{2m-1}(n;r_1,\ldots,r_m),$$

such that $\bar{t}_i^s([X^{2m-1},f])=0$, $i=1,2$. Then X^{2m-1} is decomposable.

The proof can be sketched as follows. Let f' be the homotopy inverse of f. By assumption, we can modify f' to make $f'|M^{2m-1}$ and $f'|N^{2m-1}$ be embeddings. Then

$$W=X^{2m-1}-\operatorname{int}f'(M^{2m-1})-\operatorname{int}f'(N^{2m-1})$$

is an h-cobordism of K^{2m-2} to itself. When f is a simple homotopy equivalence, we can show that W is equivalent to a product. Therefore, X^{2m-1} is decomposable.

Suppose there is an h-cobordism W of K^{2m-2} to itself such that $X^{2m-1}=M^{2m-1}\cup W\cup N^{2m-1}$. We can choose a homotopy equivalence $h:X^{2m-1}\to L^{2m-1}$ such that $h|W:W\to K^{2m-2}\times[0,1]$ is a homotopy equivalence and $h|\partial W$ is a diffeomorphism. Then $\tau(h)=i_*\tau(h|W)$ where $i_*:\mathrm{Wh}(\pi(K^{2m-2}\times[0,1]))\to\mathrm{Wh}(\pi(L^{2m-1}))$ induced by

$$i:K^{2m-2}\times[0,1]\hookrightarrow L^{2m-1}$$

and $\tau(h)$ is the torsion of h. If X^{2m-1} is decomposable $\tau(h|W)=0$. Hence $\tau(h)=0$. Thus we have proved

THEOREM 4. *Let X^{2m-1} be a closed manifold which is homotopy equivalent to $L^{2m-1}(n;r_1,\ldots,r_m)$. Suppose X^{2m-1} is decomposable, then X^{2m-1} is simple homotopy equivalent to $L^{2m-1}(n;r_1,\ldots,r_m)$.*

For $u\in 2\mathrm{Wh}(Z_n)$, there is an h-cobordism W_u of K^{2m-2} to itself such that $\tau(W_u,K)=u$. Let $X_u=M^{2m-1}\cup W_u\cup N^{2m-1}$. Let $\tilde{W}_u$ and $\tilde{X}_u$ be the universal coverings of W_u and X_u respectively. $\tilde{W}_u$ is an h-cobordism of $S^{2p-1}\times S^{2q-1}$ to itself. $S^{2p-1}\times S^{2q-1}$ is simply-connected. Then $\tilde{W}_u\simeq S^{2p-1}\times S^{2q-1}\times[0,1]$ and $\tilde{X}_u$ is a homotopy sphere supporting a free Z_n action with orbit space X_u. W_u is homotopy equivalent to $K^{2m-2}\times[0,1]$. Let $H_u:W_u\to K^{2m-2}\times[0,1]$ be a homotopy equivalence such that $H_u|\partial W_u$ is a diffeomorphism. Let $h_u=\mathrm{id}\cup H_u\cup\mathrm{id}$. Then $h_u:X_u\to L^{2m-1}$ is a homotopy equivalence and $\tau(h_u)=u$.

COROLLARY 5. *For $n\geqq 4$, there are infinitely many closed manifolds X_u^{2m-1} which are homotopy equivalent to $L^{2m-1}(n;r_1,\ldots,r_m)$ such that $t_i^h(X_u^{2m-1})=0$, $i=1,2$, but none of X_u^{2m-1} are simple homotopy equivalent to $L^{2m-1}(n;r_1,\ldots,r_m)$.*

Using the same techniques in [7] we can prove

THEOREM 6. *Let* (Σ^{2m-1}, Z_n) *be a decomposable free* Z_n *action on homotopy sphere* Σ^{2m-1}. *Then* Σ^{2m-1} *supports infinitely many inequivalent free* Z_n *actions of which the orbit spaces are of same simple homotopy type.*

REMARK 7. If $r_i = r_{p+i}$, $i = 1, \ldots, p$. Then the condition in Theorem 3 can be weakened to require that $\bar{t}_2^s([X^{2m-1}, f]) = 0$.

REMARK 8. Suppose n is odd. Let $A : \mathscr{S}^s(L^{2k-1}) \to C^{Z_n - 1}$ be the Atiyah-Singer invariants [5]. For $x \in [L^{2k-1}, G/O]$, let $x = \eta(a)$ for $a \in \mathscr{S}^s(L^{2k-1})$. Define $\bar{A}(x) = A(a)$. Using results in [6] it is easy to show that $\bar{A}$ is well defined. Let

$$T_1 : \mathscr{S}^s(L^{2m-1}) \xrightarrow{\eta} [L^{2m-1}, G/O] \to [L^{2p-1}, G/O] \xrightarrow{\bar{A}} C^{Z_n - 1}$$

$$T_2 : \mathscr{S}^s(L^{2m-1}) \xrightarrow{\eta} [L^{2m-1}, G/O] \to [L^{2q-1}, G/O] \xrightarrow{\bar{A}} C^{Z_n - 1}.$$

Then we can replace the condition in Theorem 3 by requiring that $T_i([X^{2m-1}, f]) = 0, i = 1, 2$.

REMARK 9. Theorem 3, Theorem 4 and Theorem 5 have been independently obtained by Chao-chu Liang, who is a student of G. R. Livesay.

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