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LOCALLY NICE CODIMENSION ONE MANIFOLDS ARE LOCALLY FLAT

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ABSTRACT. The main result is that an $(n-1)$-manifold M in an n-manifold Q $(n \geqq 5)$ is locally flat provided $Q - M$ is locally simply connected at each point of M. Such a theorem has been obtained by Price-Seebeck in case M is locally flat at some point. This paper uses further application of their work to eliminate the additional hypothesis.

Since this note depends so heavily on the papers of Price and Seebeck, the reader is referred to [**4**], [**5**] for definitions of the terms used here. The author wishes to express his gratitude to L. S. Husch, J. G. Hollingsworth, and C. L. Seebeck for stimulating discussions.

The first lemma, an easy consequence of [**5**], is known to several people; its application to tori in Lemma 2, which is similar to Kirby's idea in [**2**], is the key to this paper.

LEMMA 1. *Suppose* $h: E^{n-1} \times \{0\} \to E^n$ $(n \geqq 5)$ *is an embedding such that* $E^n - h(E^{n-1} \times \{0\})$ *is* 1-*LC at each point of* $h(E^{n-1} \times \{0\})$ *and there exists a positive number D for which* $d(h, \text{incl}) < D$. *Then* $h(E^{n-1} \times \{0\})$ *is locally flat.*

PROOF. The standard contraction of E^n to the interior of the unit ball B^n shrinks $h(E^{n-1} \times \{0\})$ so that, because of the bound on the displacement of h, attaching $(E^{n-1} \times \{0\}) - \operatorname{Int} B^n$ to the image of $h(E^{n-1} \times \{0\})$ produces a manifold M (homeomorphic to E^{n-1}). Routine verification establishes that $E^n - M$ is 1-LC at each point of M, and obviously M is locally flat at points of $(E^{n-1} \times \{0\}) - B^n$. Corollary 7 of [**5**] then implies that M is locally flat, and the lemma follows.

We let T^n denote the n-dimensional torus $S^1 \times \cdots \times S^1$ (n factors).

LEMMA 2. *Suppose h is an embedding of* T^{n-1} $(n \geqq 5)$ *into* $T^{n-1} \times E^1$ *such that h is a homotopy equivalence and* $(T^{n-1} \times E^1) - h(T^{n-1})$ *is* 1-*ULC. Then* $h(T^{n-1})$ *is locally flat.*

REMARK. By appealing to [**2**, Proposition 4] we may assume that h is homotopic to the inclusion map $T^{n-1} \to T^{n-1} \times \{0\} \subset T^{n-1} \times E^1$.

PROOF. Let $p': E^{n-1} \to T^{n-1}$ and $p = p' \times 1: E^{n-1} \times E^1 = E^n \to T^{n-1} \times E^1$ denote the obvious covering maps. The existence of a homotopy

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between h and the inclusion $T^{n-1} \to T^{n-1} \times \{0\} \subset T^{n-1} \times E^1$ implies, by standard covering space arguments, not only that $p^{-1}h(T)$ is homeomorphic to $E^{n-1} \times \{0\} = p^{-1}(T^{n-1} \times \{0\})$, but also that the distance between the homeomorphism and the inclusion $E^{n-1} \times \{0\} \to E^n$ is bounded. Therefore, $p^{-1}h(T^{n-1})$ is locally flat in E^n, and $h(T^{n-1})$ must be locally flat as well.

LEMMA 3. *If T is an $(n-1)$-torus in E^n $(n \geqq 5)$ such that $E^n - T$ is 1-ULC, then T is locally flat.*

PROOF. It is known [7, Proposition 3] that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pair of disjoint δ-embeddings $h_1, h_2 \colon T \to E^n$ there are strong ε-deformation retractions of X, the closure of the region bounded by $h_1(T)$ and $h_2(T)$, onto $h_1(T)$ and $h_2(T)$. Without much extra work one also can require that the deformation r_t of X onto, say, $h_1(T)$ satisfy $r_1 h_2 = h_1$. Consider the quotient space Q^* obtained from X by identifying $h_1(x)$ with $h_2(x)$, for all points x in T. Then $\pi_1(Q^*)$ is the semidirect product of $\pi_1(X)$ and Z (the integers under addition), where the appropriate automorphism ϕ of $\pi_1(X)$ to itself is given by the composition of isomorphisms

$$\begin{tikzcd}
\pi_1(X, h_1(x_0)) \arrow[d, "(i_*)^{-1}"'] & & \pi_1(X, h_1(x_0)) \\
\pi_1(h_1(T), h_1(x_0)) \arrow[r, "(h_2 h_1^{-1})_*"] & \pi_1(h_2(T), h_2(x_0)) \arrow[r, "(r_1)_*"] & \pi_1(h_1(T), h_1(x_0)). \arrow[u, "i_*"']
\end{tikzcd}$$

Then $(r_1 h_2 h_1{}^{-1})_*[\gamma] = [r_1(\alpha) * \gamma * (r_1(\alpha))^{-1}]$, where α denotes the path from $h_1(x_0)$ to $h_2(x_0)$ that follows the track of the deformation of X to $h_2(T)$ juxtaposed with the path following the track of $h_2(x_0)$ under the deformation r_t. Consequently, by choosing ε sufficiently small, we can force α to be so small that $r_1(\alpha)$ is contained in a cell in $h_1(T)$, which implies that the automorphism ϕ of $\pi_1(X)$ is the identity. Thus, we shall assume, simply by imposing such restrictions on the displacement of h_1 and h_2, that the fundamental group of the quotient space Q^* so constructed is the direct product

$$\pi_1(Q^*) \cong \pi_1(X) \times Z \cong \pi_1(T) \times Z.$$

Let W_i $(i = 1, 2)$ denote the components of $E^n - T$. Seebeck [7, Lemma 8] has established the existence of arbitrarily small PL-homeomorphisms θ_1, θ_2 of E^n to itself such that $\theta_i(T) \subset W_i$. Thinking of $\theta_i | T$ as h_i, we form X and Q^* as before and note that in this case Q^* is a closed n-manifold, since some neighborhood of the seam in Q^* is homeomorphic to a neighborhood of T in E^n. Our goal is to locate a neighborhood of T homeomorphic to $T^{n-1} \times E^1$ by proving that Q^* is homeomorphic to T^n. As indicated in the preceding paragraph, we can require θ_i $(i = 1, 2)$ to be so close to the identity that

$$\pi_1(Q^*) \cong \pi_1(T) \times Z \cong \pi_1(T^n).$$

Let X_i denote the closure of the region bounded by T and $h_i(T) = \theta_i(T)$, and let θ_i^k denote the composition of k applications of θ_i $(i = 1, 2)$. Then Q^* is covered by the subset of E^n

$$U = \left(\bigcup_{k=1}^{\infty} \theta_1^k(X_1) \right) \cup \left(\bigcup_{k=1}^{\infty} \theta_2^k(X_2) \right),$$

which deformation retracts to both X and T. Thus

$$\pi_i(Q) \cong \pi_i(U) \cong \pi_i(T) = 0 \qquad (i > 1).$$

By [**3**] Q^* is homotopy equivalent to a finite complex K (invocation of [**3**] is not necessary: It is quite easy to describe a triangulation of Q^* inherited from E^n), and since K and T^n are $K(\pi, 1)$'s with isomorphic fundamental groups, Q^* is then homotopy equivalent to T^n. By [**1**] Q^* is homeomorphic to T^n. Certainly the covering space U of Q^* can be viewed as a covering corresponding to the subgroup $\pi_1(T)$ in the decomposition of $\pi_1(Q) \cong \pi_1(T) \times Z$. But now it follows from [**2**, Proposition 4] that $T^{n-1} \times E^1$ is also such a covering, so U is topologically equivalent to $T^{n-1} \times E^1$. Finally, application of Lemma 2 implies that T is locally flat in $U \subset E^n$.

THEOREM 4. *Let M be an $(n - 1$-manifold topologically embedded in an n-manifold Q $(n \geqq 5)$ such that $Q - M$ is 1-LC at each point of M. Then M is locally flat.*

PROOF. Fix a point p of M, and consider an open n-cell W in Q containing p. There is an embedding f of $T^{n-2} \times I$ into M such that $p \in f(T^{n-2} \times \operatorname{Int} I)$ and $f(T^{n-2} \times I) \subset W$.

After describing deformation retractions of finitely many nested neighborhoods of $f(T^{n-2} \times \operatorname{Int} I)$ to $f(T^{n-2} \times \operatorname{Int} I)$ in $Q - f(T^{n-2} \times \partial I)$, we use infinite radial engulfing techniques similar to those of [**5**] and [**7**], as suggested in [**6**], to prove that there exists a homeomorphism G of W onto itself such that

(i) $G | f(T^{n-2} \times \partial I) = \text{identity}$,

(ii) $Gf(T^{n-2} \times \operatorname{Int} I) \cap f(T^{n-2} \times \operatorname{Int} I) = \varnothing$.

Hence, $Gf(T^{n-2} \times I) \cup f(T^{n-2} \times I)$ is an $(n - 1)$-torus T. Using the fact that G is a homeomorphism of W, one can establish by elementary methods that $W - T$ is 1-LC at each point of T. Then Lemma 3 implies that T is locally flat in $W \subset Q$. Since T contains a neighborhood of p relative to M, M is locally flat at p.

COROLLARY 5. *Suppose M is an $(n-1)$-manifold in an n-manifold Q $(n \geqq 5)$ and U is an open subset of M such that $Q - M$ is* 1-*LC at each point of U. Then M is locally flat at each point of U.*

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