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LOCAL SPECTRAL MAPPING THEOREMS

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This note is concerned with various "localized versions" of the spectral mapping theorem (cf. [**3**, VII.3.11]) for bounded linear operators in a complex Banach space. The details of the proof and some additional results will be published elsewhere.

Let X be a complex Banach space and let $T \in B(X)$, the Banach algebra of all bounded linear operators on X. We recall (cf. [**2**, p. 1], [**3**, p. 1931]) that if T has the single-valued extension property then there exist a maximal open set $\rho_T(x)$ containing $\rho(T)$ and a unique holomorphic function $\tilde{x}_T : \rho_T(x) \to X$ such that $(\lambda I - T)\tilde{x}_T(\lambda) = x$ for all $\lambda \in \rho_T(x)$. The complementary set $\sigma_T(x) = \boldsymbol{C} - \rho_T(x)$, which we call the *local spectrum of* x (with respect to T), is compact and is contained in $\sigma(T)$, the spectrum of T. If $F \subseteq \boldsymbol{C}$ is closed we introduce the spectral manifold $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$.

THEOREM 1. *Let f be holomorphic on a neighborhood of $\sigma(T)$ and suppose that T and $f(T)$ have the single-valued extension property. Then $f(\sigma_T(x)) = \sigma_{f(T)}(x)$ for all $x \in X$.*

This result[2] was proved by the second-named author in his dissertation [**4**]. The proof given there is similar to the proof of a theorem due to Colojoară and Foiaş ([**1**], [**2**, p. 71]). In fact, the conclusion of Theorem 1 is equivalent to the condition that if F is a closed subset of $\sigma(f(T)) = f(\sigma(T))$, then

$$X_{f(T)}(F) = X_T(f^{-1}(F)).$$

If $T \in B(X)$ and Y is a (closed) subspace of X which is invariant under T, then the spectrum of the restriction $T|Y$ may be either smaller or larger than $\sigma(T)$. It is often desirable to limit attention to subspaces which do not increase the spectrum under restriction (e.g., ultra-invariant subspaces).

THEOREM 2. *Let Y be a subspace invariant under T such that $\sigma(T|Y) \subseteq \sigma(T)$ and let f be holomorphic on a neighborhood of $\sigma(T)$. Then Y is invariant*

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[2] ADDED IN PROOF. After this note was communicated we discovered this theorem with a similar proof in C. Apostol, *Teorie spectrală şi calcul funcţional*, Stud. Cerc. Mat. **20**, no. 5 (1968), 635–668.

under $f(T)$, $f(T)|Y = f(T|Y)$, *and*

$$f(\sigma(T|Y)) = \sigma(f(T|Y)) = \sigma(f(T)|Y) \subseteq f(\sigma(T)).$$

If Y is an invariant subspace for T such that $\sigma(T|Y) \subseteq \sigma(T)$ and if $y \in Y$, then $\sigma_T(y) \subseteq \sigma_{T|Y}(y)$, but this inclusion can be proper. The next result holds for subspaces for which equality holds (e.g., spectral maximal spaces).

THEOREM 3. *Suppose that* Y *is a subspace invariant under* T *such that* $\sigma_{T|Y}(y) = \sigma_T(y)$ *for all* $y \in Y$. *Let* f *be holomorphic on a neighborhood of* $\sigma(T)$ *and let* T *and* $f(T)$ *have the single-valued extension property. If* $y \in Y$, *we have*

$$\sigma_{f(T)|Y}(y) = \sigma_{f(T|Y)}(y) = f(\sigma_{T|Y}(y)) = f(\sigma_T(y)) = \sigma_{f(T)}(y).$$

If $T \in B(X)$ and Y is a (closed) subspace of X which is invariant under T and if T/Y is the operator in the quotient space X/Y defined by $(T/Y)[x] = [Tx]$, then the spectrum of T/Y may be either smaller or larger than $\sigma(T)$. It is often desirable to limit attention to subspaces which do not increase the spectrum under quotients (e.g., ultra-invariant subspaces).

THEOREM 4. *Let* Y *be a subspace invariant under* T *such that* $\sigma(T/Y) \subseteq \sigma(T)$ *and let* f *be holomorphic on a neighborhood of* $\sigma(T)$. *Then* Y *is invariant under* $f(T)$, $f(T)/Y = f(T/Y)$, *and*

$$f(\sigma(T/Y)) = \sigma(f(T/Y)) = \sigma(f(T)/Y) \subseteq f(\sigma(T)).$$

In addition, if T, $f(T)$, T/Y, *and* $f(T)/Y$ *have the single-valued extension property and* $x \in X$, *then we have*

$$\sigma_{f(T)/Y}([x]) = \sigma_{f(T/Y)}([x]) = f(\sigma_{T/Y}([x])) \subseteq f(\sigma_T(x)) = \sigma_{f(T)}(x).$$

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