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PERIODIC AND HOMOGENEOUS STATES ON A VON NEUMANN ALGEBRA. II[1]

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This paper is a natural continuation of the previous paper [**9**]. In [**9**], we proved a structure theorem for a von Neumann algebra with a fixed periodic and homogeneous state. In this paper, we will show that the structure theorem in [**9**] determines intrinsically the algebraic type of a factor with a periodic and *inner* homogeneous state (see Definition 1). We keep the terminologies and the notations in [**9**].

DEFINITION 1. A normal state φ on a von Neumann algebra $\mathscr{M}$ is said to be *inner homogeneous* if $G(\varphi) \cap \operatorname{Int}(\mathscr{M})$ acts ergodically on $\mathscr{M}$, that is, if the group of all inner automorphisms of $\mathscr{M}$ leaving φ invariant has no fixed points other than the scalar multiples of the identity.

For each $a \in \mathscr{M}$, we write

$$\operatorname{Ad}(a)x = axa^*, \qquad x \in \mathscr{M}.$$

Since $\operatorname{Ad}(u) \in G(\varphi)$ for a unitary $u \in \mathscr{M}$ if and only if u falls in $\mathscr{M}_\varphi$, the centralizer of φ, the inner homogeneity of φ is equivalent to the fact that $\mathscr{M}'_\varphi \cap \mathscr{M} = \{\lambda 1\}$. Hence $\mathscr{M}_\varphi$ is a II_1-factor and $\mathscr{M}$ itself is also a factor.

We consider two periodic and inner homogeneous faithful normal states φ and ψ on $\mathscr{M}$. We denote by $\{\mathscr{M}_n^\varphi : n = 0, \pm 1, \ldots\}$ and $\{\mathscr{M}_n^\psi : n = 0, \pm 1, \ldots\}$ the decompositions of $\mathscr{M}$ in [**9**, Theorem 11] corresponding to φ and ψ respectively. By [**9**, Theorem 13], φ and ψ have the same period, say $T > 0$. Let $\kappa = e^{-2\pi/T}$, $0 < \kappa < 1$.

Following Connes' idea, we consider the tensor product $\mathscr{P} = \mathscr{M} \otimes \mathscr{L}(\mathfrak{H}_2)$ of $\mathscr{M}$ and the 2×2-matrix algebra $\mathscr{L}(\mathfrak{H}_2)$. Let $\{e_{i,j}: i, j = 1, 2\}$ be a system of matrix units in $\mathscr{L}(\mathfrak{H}_2)$. Every $x \in \mathscr{P}$ is of the form

$$x = x_{11} \otimes e_{11} + x_{12} \otimes e_{12} + x_{21} \otimes e_{21} + x_{22} \otimes e_{22},$$

where $x_{ij} \in \mathscr{M}$. We define a faithful state χ on $\mathscr{P}$ by

$$\chi(x) = \tfrac{1}{2}(\varphi(x_{11}) + \psi(x_{22})).$$

Connes showed in [**3**] that there exists a strongly continuous one-

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parameter family $\{u_t\}$ of unitaries in $\mathscr{M}$ such that $\sigma_t^\chi(1 \otimes e_{12}) = u_t \otimes e_{12}$ and $\sigma_t^\psi(x) = u_t^* \sigma_t^\varphi(x) u_t$, $x \in \mathscr{M}$.

LEMMA 2. $u_{s+t} = \sigma_s^\varphi(u_t)u_s$.[2]

PROOF. The above cocycle equality follows from the simple calculation.

$$\begin{aligned} u_{s+t} \otimes e_{12} &= \sigma_{s+t}^\chi(1 \otimes e_{12}) = \sigma_s^\chi \cdot \sigma_t^\chi(1 \otimes e_{12}) \\ &= \sigma_s^\chi(u_t \otimes e_{12}) = \sigma_s^\chi((u_t \otimes e_{11})(1 \otimes e_{12})) \\ &= (\sigma_s^\varphi(u_t) \otimes e_{11})(u_s \otimes e_{12}) = \sigma_s^\varphi(u_t)u_s \otimes e_{12}. \end{aligned}$$

Since $\sigma_T^\varphi = \sigma_T^\psi = \mathrm{id}$, u_T must be a scalar multiple of 1, so that one can find $1 \leqq \alpha < e^{2\pi/T}$ with $u_T = \alpha^{iT}1$. Let $v_t = \alpha^{-it}u_t$. We have then the following properties:

(1) $$v_t^* \sigma_t^\varphi(x) v_t = \sigma_t^\psi(x), \qquad x \in \mathscr{M};$$

(2) $$v_{s+t} = \sigma_s^\varphi(v_t)v_s = \sigma_t^\varphi(v_s)v_t;$$

(3) $$v_T = 1.$$

We define a one-parameter family $\{\rho_t\}$ of isometries of $\mathscr{M}$ onto $\mathscr{M}$ by

(4) $$\rho_t(x) = \sigma_t^\varphi(x)v_t, \qquad x \in \mathscr{M}.$$

From (2) and (3) we obtain the following:

(5) $$\rho_{s+t} = \rho_s \cdot \rho_t;$$

(6) $$\rho_T = \mathrm{id}.$$

For each integer n, let $\mathscr{V}_n = \{x \in \mathscr{M} : \rho_t(x) = \kappa^{int}x\}$. Since ρ_t has period T, $\mathscr{V}_n \neq \{0\}$ for some n.

LEMMA 3. *An $x \in \mathscr{M}$ falls in $\mathscr{V}_n$ if and only if*

$$\sigma_t^\chi(x \otimes e_{12}) = \alpha^{it}\kappa^{int}(x \otimes e_{12}).$$

PROOF. We compute as follows:

$$\begin{aligned} \sigma_t^\chi(x \otimes e_{12}) &= \sigma_t^\chi((x \otimes e_{11})(1 \otimes e_{12})) \\ &= (\sigma_t^\varphi(x) \otimes e_{11})(u_t \otimes e_{12}) = \sigma_t^\varphi(x)u_t \otimes e_{12} \\ &= \alpha^{it}\sigma_t(x)v_t \otimes e_{12} = \alpha^{it}\rho_t(x) \otimes e_{12}. \end{aligned}$$

Thus, the assertion follows.

Therefore, we conclude that $x \in \mathscr{V}_n$ if and only if $\alpha\kappa^n\chi(y(x \otimes e_{12}))$

[2] ADDED IN PROOF. This cocycle equation is mentioned in the final version of [3], which was missed from an earlier version and not available at the time when this article was finished.

$= \chi((x \otimes e_{12})y)$ for every $y \in \mathscr{P}$.

LEMMA 4. *If $x \in \mathscr{V}_n$ and $y \in \mathscr{M}$, then we have*

$$\alpha\kappa^n\psi(yx) = \varphi(xy). \tag{7}$$

PROOF. For each $x \in \mathscr{V}_n$ and $y \in \mathscr{M}$, we have

$$\begin{aligned} \alpha\kappa^n\psi(yx) &= 2\alpha\kappa^n\chi((y \otimes e_{21})(x \otimes e_{12})) \\ &= 2\chi((x \otimes e_{12})(y \otimes e_{21})) \quad \text{by Lemma 3} \\ &= \varphi(xy). \end{aligned}$$

LEMMA 5. $\mathscr{M}_m^\varphi \mathscr{V}_l \mathscr{M}_n^\psi \subset \mathscr{V}_{m+l+n}$.

PROOF. For each $a \in \mathscr{M}_m^\phi$, $b \in \mathscr{M}_n^\psi$ and $x \in \mathscr{V}_l$, we have

$$\begin{aligned} \rho_t(axb) &= \sigma_t^\varphi(axb)v_t = \kappa^{imt}a\sigma_t^\varphi(x)\sigma_t^\varphi(b)v_t \\ &= \kappa^{imt}a\sigma_t^\varphi(x)v_tv_t^*\sigma_t^\varphi(b)v_t \\ &= \kappa^{imt}a\kappa^{ilt}x\kappa^{int}b = \kappa^{i(m+l+n)t}axb. \end{aligned}$$

Since $\mathscr{M}_m^\varphi$, $m \geqq 1$, contains isometries and $\mathscr{M}_n^\psi$, $n \leqq -1$, contains co-isometries by [**9**], we conclude that $\mathscr{V}_n \neq \{0\}$ for every integer n. Since $\mathscr{M}_0^\varphi$ and $\mathscr{M}_0^\psi$ are both factors, we conclude that, for every pair of nonzero projections $p \in \mathscr{M}_0^\varphi$ and $q \in \mathscr{M}_0^\psi$,

$$p\mathscr{V}_n q \neq \{0\}. \tag{8}$$

For an $x \in \mathscr{V}_n$, let $x = uh = ku$ be the right and left polar decomposition of x. We have then

$$k(\kappa^{int}u) = \kappa^{int}x = \rho_t(x) = \sigma_t^\varphi(x)v_t = \sigma_t^\varphi(k)\sigma_t^\varphi(u)v_t.$$

By the unicity of the polar decomposition, we get $k = \sigma_t^\varphi(k)$ and $\kappa^{int}u = \rho_t(u)$; hence $u \in \mathscr{V}_n$. Similarly, we get $h = \sigma_t^\psi(h)$.

Thus, we obtain the following:

LEMMA 6. *For every $x \in \mathscr{V}_n$, we have*

$$xx^* \in \mathscr{M}_0^\varphi \quad \textit{and} \quad x^*x \in \mathscr{M}_0^\psi.$$

If u is a partial isometry in $\mathscr{V}_n$, then we have, by (7),

$$\varphi(uxu^*) = \alpha\kappa^n\psi(xu^*u), \qquad x \in \mathscr{M}.$$

Taking fact (8) into account and making use of the usual exhaustion arguments, we conclude that there exists an isometry or a co-isometry u in $\mathscr{V}_n$ according as $\alpha\kappa^n \leqq 1$ or $\alpha\kappa^n > 1$. Considering various n, we conclude the following:

THEOREM 7. *For periodic inner homogeneous faithful states* φ *and* ψ *on a factor* $\mathscr{M}$, *there exist isometries* u *and* v *in* $\mathscr{M}$ *such that*

$$\psi(x) = \varphi(uxu^*)/\varphi(uu^*), \qquad (9)$$

$$\varphi(x) = \psi(vxv^*)/\psi(vv^*), \qquad x \in \mathscr{M};$$

$$uu^* \in \mathscr{M}_0^{\varphi} \quad \text{and} \quad vv^* \in \mathscr{M}_0^{\psi}. \qquad (10)$$

Fixing a periodic inner homogeneous state φ on a factor $\mathscr{M}$, for each projection $p \in \mathscr{M}_0^{\varphi}$ we define a state φ_p on $p\mathscr{M}p$ by

$$\varphi_p(x) = \varphi(x)/\varphi(p), \qquad x \in p\mathscr{M}p.$$

By Theorem 7, any other periodic inner homogeneous state ψ on $\mathscr{M}$ is unitarily equivalent to φ_p for some $p \in \mathscr{M}_0^{\varphi}$; more precisely, there exists an isometry $u \in \mathscr{M}$ with $uu^* = p \in \mathscr{M}_0^{\varphi}$ such that $\psi(x) = \varphi_p(uxu^*)$ for every $x \in \mathscr{M}$. Thus, the set $\{\varphi_p : p \in \mathscr{M}_0^{\varphi}\}$ exhausts all possible periodic inner homogeneous states up to unitary equivalence.

THEOREM 8. *Let* p *and* q *be two projections in* $\mathscr{M}_0^{\varphi}$, *and let* u *and* v *be isometries in* $\mathscr{M}$ *with* $uu^* = p$ *and* $vv^* = q$. *Let* $\psi(x) = \varphi_p(uxu^*)$ *and* $\omega(x) = \varphi_q(vxv^*)$, $x \in \mathscr{M}$. *Then* ψ *and* ω *are unitarily equivalent if and only if* $\varphi(p) = \kappa^n\varphi(q)$ *for some integer* n.

THEOREM 9. *For the state* ψ *defined in the previous theorem, there exists* $\sigma \in \mathrm{Aut}(\mathscr{M})$ *with* $\psi = \varphi \circ \sigma$ *if and only if there exists an isomorphism* ρ *of* $\mathscr{M}_0^{\varphi}$ *onto* $p\mathscr{M}_0^{\varphi}p$ *and a partial isometry* w *in* $\mathscr{M}_0^{\varphi}$ *such that* $\rho \cdot \theta(x) = w\theta \cdot \rho(x)w^*$, $x \in \mathscr{M}_0^{\varphi}$, *where* θ *denotes the isomorphism described in* [**9**, *Theorem* 11].

By the following theorem, one can distinguish the algebraic type of $\mathscr{M}$ in terms of $\{\mathscr{M}_0^{\varphi}, \theta\}$.

THEOREM 10. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be factors equipped with periodic inner homogeneous states* φ *and* ψ *respectively. Let* $\{\mathscr{M}_0, \theta\}$ *and* $\{\mathscr{N}_0, \rho\}$ *be the relevant couples of* II_1*-factors and isomorphisms described in* [**9**, *Theorem* 11] *respectively. Let* $e_{-1} = \theta(1)$ *and* $f_{-1} = \rho(1)$. *Necessary and sufficient conditions for* $\mathscr{M}$ *and* $\mathscr{N}$ *to be isomorphic are that* (i) $\varphi(e_{-1}) = \psi(f_{-1})$; (ii) *there exists an isomorphism* σ *of* $\mathscr{N}_0$ *onto* $p\mathscr{M}_0p$ *for some projection* p *with* $p \geqq e_{-1}$ *and a partial isometry* w *in* $\mathscr{M}_0$ *such that* $w\theta \cdot \sigma(x)w^* = \sigma \cdot \rho(x)$, $x \in \mathscr{N}_0$.

Making use of the new results of Connes in [**5**], we can prove the following:

THEOREM 11. *Let* $\mathscr{M}$ *be a factor equipped with a periodic homogeneous state. The existence of a periodic inner homogeneous state on* $\mathscr{M}$ *is equiva-*

lent to the fact that $S(\mathscr{M}) \neq \{0, 1\}$, *where* $S(\mathscr{M})$ *means the invariant of* $\mathscr{M}$ *defined by Connes in* [**1**].

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