PERIODIC AND HOMOGENEOUS STATES ON A VON NEUMANN ALGEBRA. II

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This paper is a natural continuation of the previous paper [9]. In [9], we proved a structure theorem for a von Neumann algebra with a fixed periodic and homogeneous state. In this paper, we will show that the structure theorem in [9] determines intrinsically the algebraic type of a factor with a periodic and inner homogeneous state (see Definition 1). We keep the terminologies and the notations in [9].

DEFINITION 1. A normal state \( \varphi \) on a von Neumann algebra \( \mathcal{M} \) is said to be inner homogeneous if \( G(\varphi) \cap \text{Int}(\mathcal{M}) \) acts ergodically on \( \mathcal{M} \), that is, if the group of all inner automorphisms of \( \mathcal{M} \) leaving \( \varphi \) invariant has no fixed points other than the scalar multiples of the identity.

We consider two periodic and inner homogeneous faithful normal states \( \varphi \) and \( \psi \) on \( \mathcal{M} \). We denote by \( \{\mathcal{M}^*_{\varphi}: n = 0, \pm 1, \ldots\} \) and \( \{\mathcal{M}^*_{\psi}: n = 0, \pm 1, \ldots\} \) the decompositions of \( \mathcal{M} \) in [9, Theorem 11] corresponding to \( \varphi \) and \( \psi \) respectively. By [9, Theorem 13], \( \varphi \) and \( \psi \) have the same period, say \( T > 0 \). Let \( \kappa = e^{-2\pi/T}, 0 < \kappa < 1 \).

Following Connes’ idea, we consider the tensor product \( \mathcal{P} = \mathcal{M} \otimes \mathcal{L}(\mathcal{S}_2) \) of \( \mathcal{M} \) and the \( 2 \times 2 \)-matrix algebra \( \mathcal{L}(\mathcal{S}_2) \). Let \( \{e_{ij}: i, j = 1, 2\} \) be a system of matrix units in \( \mathcal{L}(\mathcal{S}_2) \). Every \( x \in \mathcal{P} \) is of the form

\[
x = x_{11} \otimes e_{11} + x_{12} \otimes e_{12} + x_{21} \otimes e_{21} + x_{22} \otimes e_{22},
\]

where \( x_{ij} \in \mathcal{M} \). We define a faithful state \( \chi \) on \( \mathcal{P} \) by

\[
\chi(x) = \frac{1}{2}(\varphi(x_{11}) + \psi(x_{22})).
\]

Connes showed in [3] that there exists a strongly continuous one-

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parameter family \( \{ u_t \} \) of unitaries in \( \mathcal{M} \) such that \( \sigma_t^x(1 \otimes e_{12}) = u_t \otimes e_{12} \) and \( \sigma_t^x(x) = u_t^* \sigma_t^x(x) u_t, \ x \in \mathcal{M} \).

**Lemma 2.** \( u_{s+t} = \sigma_s^x(u_t) u_s \).\(^2\)

**Proof.** The above cocycle equality follows from the simple calculation.

\[
u_{s+t} \otimes e_{12} = \sigma_{s+t}^x(1 \otimes e_{12}) = \sigma_s^x \cdot \sigma_t^x(1 \otimes e_{12}) = \sigma_s^x(u_t \otimes e_{12}) = \sigma_s^x((u_t \otimes e_{11})(1 \otimes e_{12})) = (\sigma_s^x(u_t) \otimes e_{11})(u_s \otimes e_{12}) = \sigma_s^x(u_t) u_s \otimes e_{12}.
\]

Since \( \sigma_T^x = \sigma_0^x = \text{id} \), \( u_T \) must be a scalar multiple of 1, so that one can find \( 1 \leq x < e^{2\pi/T} \) with \( u_T = x^T 1 \). Let \( v_i = x^{-i} u_i \). We have then the following properties:

1. \( v^*_i \sigma_t^x(v_i) = \sigma_t^y(x), \ x \in \mathcal{M} \);
2. \( v_{s+t} = \sigma_s^y(v_t) v_s = \sigma_s^y(v_s)v_t \);
3. \( v_T = 1 \).

We define a one-parameter family \( \{ \rho_t \} \) of isometries of \( \mathcal{M} \) onto \( \mathcal{M} \) by

\[
\rho_t(x) = \sigma_t^x(x)v_t, \ x \in \mathcal{M}.
\]

From (2) and (3) we obtain the following:

5. \( \rho_{s+t} = \rho_s \cdot \rho_t \);
6. \( \rho_T = \text{id} \).

For each integer \( n \), let \( \mathcal{Y}_n = \{ x \in \mathcal{M} : \rho_t(x) = \kappa^{nt} x \} \). Since \( \rho_t \) has period \( T \), \( \mathcal{Y}_n \neq \{ 0 \} \) for some \( n \).

**Lemma 3.** An \( x \in \mathcal{M} \) falls in \( \mathcal{Y}_n \) if and only if

\[
\sigma_t^x(x \otimes e_{12}) = x^t \kappa^{nt}(x \otimes e_{12}).
\]

**Proof.** We compute as follows:

\[
\sigma_t^x(x \otimes e_{12}) = \sigma_t^x((x \otimes e_{11})(1 \otimes e_{12})) = (\sigma_t^x(x) \otimes e_{11})(u_t \otimes e_{12}) = x^t \sigma_t^x(x) v_t \otimes e_{12} = x^t \rho_t(x) \otimes e_{12}.
\]

Thus, the assertion follows.

Therefore, we conclude that \( x \in \mathcal{Y}_n \) if and only if \( x \kappa^{nt}(x \otimes e_{12}) \)

\(^2\)Added in proof. This cocycle equation is mentioned in the final version of [3], which was missed from an earlier version and not available at the time when this article was finished.
= \chi((x \otimes e_{12})y) for every \( y \in \mathcal{P} \).

**Lemma 4.** If \( x \in \mathcal{V}_n \) and \( y \in \mathcal{M} \), then we have

\[
\alpha \kappa^n \psi(yx) = \varphi(xy).
\]

**Proof.** For each \( x \in \mathcal{V}_n \) and \( y \in \mathcal{M} \), we have

\[
\alpha \kappa^n \psi(yx) = 2 \alpha \kappa^n \chi((y \otimes e_{21})(x \otimes e_{12})) = 2\chi((x \otimes e_{12})(y \otimes e_{21})) \text{ by Lemma 3}
\]

\[
= \varphi(xy).
\]

**Lemma 5.** \( \mathcal{M}_m^\alpha \mathcal{V}_1 \mathcal{M}_n^\psi \subset \mathcal{V}_{m+1+n} \).

**Proof.** For each \( a \in \mathcal{M}_m^\alpha \), \( b \in \mathcal{M}_n^\psi \) and \( x \in \mathcal{V}_1 \), we have

\[
\rho_i(axb) = \sigma_i^a(axb)v_i = \kappa^{int}a\sigma_i^a(x)\sigma_i^a(b)v_i
\]

\[
= \kappa^{int}a\sigma_i^a(x)v_i v_i^* \sigma_i^a(b)v_i
\]

\[
= \kappa^{int}a\kappa^{int}x \kappa^{int}b = \kappa^{(m+1+n)tr}axb.
\]

Since \( \mathcal{M}_m^\alpha \), \( m \geq 1 \), contains isometries and \( \mathcal{M}_n^\psi \), \( n \leq -1 \), contains co-isometries by [9], we conclude that \( \mathcal{V}_n \neq \{0\} \) for every integer \( n \). Since \( \mathcal{M}_0^\alpha \) and \( \mathcal{M}_0^\psi \) are both factors, we conclude that, for every pair of nonzero projections \( p \in \mathcal{M}_0^\alpha \) and \( q \in \mathcal{M}_0^\psi \),

\[
(8) \quad pV_nq \neq \{0\}.
\]

For an \( x \in \mathcal{V}_n \), let \( x = uh = ku \) be the right and left polar decomposition of \( x \). We have then

\[
k(k^{int}u) = k^{int}x = \rho_i(x) = \sigma_i^a(x)v_i = \sigma_i^a(k)\sigma_i^a(u)v_i.
\]

By the unicity of the polar decomposition, we get \( k = \sigma_i^a(k) \) and \( k^{int}u = \rho_i(u) \); hence \( u \in \mathcal{V}_n \). Similarly, we get \( h = \sigma_i^a(h) \).

Thus, we obtain the following:

**Lemma 6.** For every \( x \in \mathcal{V}_n \), we have

\[
xx^* \in \mathcal{M}_0^\alpha \quad \text{and} \quad x^*x \in \mathcal{M}_0^\psi.
\]

If \( u \) is a partial isometry in \( \mathcal{V}_n \), then we have, by (7),

\[
\varphi(uxu^*) = \alpha \kappa^n \psi(uxu^*), \quad x \in \mathcal{M}.
\]

Taking fact (8) into account and making use of the usual exhaustion arguments, we conclude that there exists an isometry or a co-isometry \( u \) in \( \mathcal{V}_n \) according as \( \alpha \kappa^n \leq 1 \) or \( \alpha \kappa^n > 1 \). Considering various \( n \), we conclude the following:
Theorem 7. For periodic inner homogeneous faithful states \( \varphi \) and \( \psi \) on a factor \( \mathcal{M} \), there exist isometries \( u \) and \( v \) in \( \mathcal{M} \) such that

\[
\psi(x) = \varphi(uxu^*)/\varphi(uu^*),
\]
\[
\varphi(x) = \psi(vxv^*)/\psi(vv^*), \quad x \in \mathcal{M};
\]

(9) \( uu^* \in \mathcal{M}_0^0 \) and \( vv^* \in \mathcal{M}_0^0 \).

Fixing a periodic inner homogeneous state \( \varphi \) on a factor \( \mathcal{M} \), for each projection \( p \in \mathcal{M}_0^0 \) we define a state \( \varphi_p \) on \( p\mathcal{M}p \) by

\[
\varphi_p(x) = \varphi(x)/\varphi(p), \quad x \in p\mathcal{M}p.
\]

By Theorem 7, any other periodic inner homogeneous state \( \varphi \) on \( \mathcal{M} \) is unitarily equivalent to \( \varphi_p \) for some \( p \in \mathcal{M}_0^0 \); more precisely, there exists an isometry \( u \in \mathcal{M} \) with \( uu^* = p \in \mathcal{M}_0^0 \) such that \( \psi(x) = \varphi_p(uxu^*) \) for every \( x \in \mathcal{M} \). Thus, the set \( \{ \varphi_p : p \in \mathcal{M}_0^0 \} \) exhausts all possible periodic inner homogeneous states up to unitary equivalence.

Theorem 8. Let \( p \) and \( q \) be two projections in \( \mathcal{M}_0^0 \), and let \( u \) and \( v \) be isometries in \( \mathcal{M} \) with \( uu^* = p \) and \( vv^* = q \). Let \( \psi(x) = \varphi_p(uxu^*) \) and \( \omega(x) = \varphi_q(vxv^*) \), \( x \in \mathcal{M} \). Then \( \psi \) and \( \omega \) are unitarily equivalent if and only if \( \varphi(p) = \kappa^n \varphi(q) \) for some integer \( n \).

Theorem 9. For the state \( \psi \) defined in the previous theorem, there exists \( \sigma \in \text{Aut}(\mathcal{M}) \) with \( \psi = \varphi \circ \sigma \) if and only if there exists an isomorphism \( \rho \) of \( \mathcal{M}_0^0 \) onto \( p\mathcal{M}_0^0 p \) and a partial isometry \( w \) in \( \mathcal{M}_0^0 \) such that \( \rho \cdot \theta(x) = w\theta \cdot \rho(x)w^*, \quad x \in \mathcal{M}_0^0 \), where \( \theta \) denotes the isomorphism described in [9, Theorem 11].

By the following theorem, one can distinguish the algebraic type of \( \mathcal{M} \) in terms of \( \{ \mathcal{M}_0^0, \theta \} \).

Theorem 10. Let \( \mathcal{M} \) and \( \mathcal{N} \) be factors equipped with periodic inner homogeneous states \( \varphi \) and \( \psi \) respectively. Let \( \{ \mathcal{M}_0, \theta \} \) and \( \{ \mathcal{N}_0, \rho \} \) be the relevant couples of \( \Pi_1 \)-factors and isomorphisms described in [9, Theorem 11] respectively. Let \( e_{-1} = \theta(1) \) and \( f_{-1} = \rho(1) \). Necessary and sufficient conditions for \( \mathcal{M} \) and \( \mathcal{N} \) to be isomorphic are that (i) \( \varphi(e_{-1}) = \psi(f_{-1}) \); (ii) there exists an isomorphism \( \sigma \) of \( \mathcal{N}_0 \) onto \( p\mathcal{M}_0 p \) for some projection \( p \) with \( p \geq e_{-1} \) and a partial isometry \( w \) in \( \mathcal{M}_0^0 \) such that \( w\theta \cdot \sigma(x)w^* = \sigma \cdot \rho(x), \quad x \in \mathcal{N}_0 \).

Making use of the new results of Connes in [5], we can prove the following:

Theorem 11. Let \( \mathcal{M} \) be a factor equipped with a periodic homogeneous state. The existence of a periodic inner homogeneous state on \( \mathcal{M} \) is equiva-
lent to the fact that \( S(\mathcal{M}) \neq \{0, 1\} \), where \( S(\mathcal{M}) \) means the invariant of \( \mathcal{M} \) defined by Connes in [1].

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**References**


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