BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, March 1973

# EUCLIDEAN SUBRINGS OF GLOBAL FIELDS[1]

BY CLIFFORD S. QUEEN

Communicated by Hyman Bass, October 20, 1972

1. **Introduction.** The purpose of this note is to announce some results regarding the existence of euclidean subrings of global fields.

We first state the problem and give its history. Let $F$ be a global field. So $F$ is a finite extension of the rational numbers $Q$ or $F$ is a function field of one variable over a finite field $k$, where $k$ is algebraically closed in $F$. Let $S$ be a finite nonempty set of prime divisors of $F$ such that $S$ includes all infinite (i.e., archimedean) prime divisors. If $P$ is a finite (i.e., nonarchimedean) prime divisor we denote by $O_P$ its valuation ring in $F$. Now, given a finite set $S$ of the above type, we get a ring

$$O_S = \bigcap_{P \notin S} O_P$$

where $P$ ranges over all prime divisors of $F$. We note in particular that if $F$ is a number field and $S$ the set of infinite prime divisors of $F$ then $O_S$ is just the ring of $F$-integers.

It is easy to see that there always exists a finite set $S$ satisfying the above hypothesis such that $O_S$ is a unique factorization domain. Hence it seems natural to ask the following two questions:

I. Does there always exist an $S$ such that $O_S$ is a euclidean ring?

II. Can one find an algorithm on $O_S$ for suitably chosen $S$ which is related in some way to the arithmetic of the field $F$?

The history of the above two questions is as follows: In a series of articles [**1**]–[**4**] Armitage discussed I and II for function fields over arbitrary ground fields. He insisted on a choice of algorithm related to the norm from $F$ to a rational subfield. He showed that if the ground field of $F$ is infinite, then an algorithm of his spacial type was possible if and only if the genus of $F$ is zero. He also discussed the case when the ground field of $F$ is finite, but again the only situation in which he gave a positive answer to I was when $F$ is of genus zero. In [**6**], Samuel also discussed I for function fields $F$ with arbitrary fields of constants, but here also he did not get above genus zero. Finally, in [**5**], M. Madan and the present author showed that the answer to both I and II is yes for function fields of genus one over finite fields. The method in [**5**] was to specifically construct an $S$ and an algorithm on $O_S$ for given $F$.

*AMS (MOS) subject classifications* (1970). Primary 12A05; Secondary 12A05.
*Key words and phrases*. Euclid's algorithm, adeles, global fields.

[1] ADDED IN PROOF. After this announcement went to press, the author discovered that Theorem 2 was proved by O. T. O'Meara in *On the finite generation of linear groups over Hasse domains*, J. Reine Angew. Math. **217** (1965), 79–108. MR **31** #3513.

In the next section we indicate a proof that the answer to both I and II is yes for arbitrary global fields $F$. Full details of the proof and applications of the results will appear elsewhere.

2. **Results.** Let $F$ be a global field. If $P$ is a finite prime divisor of $F$ we denote by $N(P)$ the absolute norm of $P$ and we associate with each such $P$ a normalized valuation $|\ |_P$ as follows: $|O|_P = 0$ and if $x \in F - \{0\} = F^*$, then $|x|_P = N(P)^{-n}$, where $P^n$ is the power to which $P$ appears in the principal divisor $(x)$ determined by $x$ in $F$. Now if $P$ is an infinite prime divisor then $P$ corresponds to an embedding $\sigma_P$ of $F$ into the complex numbers and we determine a normalized valuation $|\ |_P$ associated to $P$ in the following way: If $\sigma_P(F)$ is a subfield of the real numbers then $|x|_P = |\sigma_P(x)|$ for all $x \in F$, where $|\ |$ is the ordinary real absolute value. Finally if $\sigma_P(F)$ is not a subfield of the reals, we set $|x|_P = |\sigma_P(x)|^2$ for all $x \in F$, where $|\ |$ is the usual complex absolute value. Hence letting $P$ range over all prime divisors of $F$, we have the well-known formula

$$\prod_P |x|_P = 1 \tag{1}$$

for all $x \in F^*$.

If $P$ is a prime divisor of $F$ we denote by $F_P$ the completion of $F$ with respect to the valuation $|\ |_P$. These fields $F_P$ are all locally compact and if $P$ is finite we denote by $R_P$ the maximal compact subring of $F_P$. We call the restricted topological product of the $F_P$ with respect to the $R_P$ the ring of adeles of $F$ and denote it by $F_A$. We further identify $F$ with its diagonal embedding in $F_A$.

Now if $F$ is a number field we denote by $S_\infty$ the set of infinite prime divisors of $F$ and if $F$ is a function field over a finite field we fix a prime divisor $P_\infty$ of $F$ and set $S_\infty = \{P_\infty\}$. Next if $x \in F^*$ we set

$$V(x) = \{\zeta \in F_A \mid |\zeta_P|_P < |x|_P \text{ for } P \in S_\infty \text{ and } |\zeta_P|_P \leqq |x|_P \text{ for } P \notin S_\infty\}.$$

THEOREM 1.

$$F_A = \bigcup_{x \in F^*} (V(x) + F).$$

INDICATION OF PROOF. If $F$ is a function field and $k$ its exact field of constants we use the Riemann-Roch theorem to choose $t \in F$ such that $F/k(t)$ is a separable extension and $|t|_{P_\infty} > 1$, with $|t|_P \leqq 1$ for all $P \neq P_\infty$. If $F$ is a number field we let $H$ denote the field of real numbers and otherwise $H$ will denote $k((t^{-1}))$, where $k((t^{-1}))$ is the quotient field of the ring of formal power series in $t^{-1}$ over $k$. Next we set $F_\infty = F \otimes_L H$, where $L = Q$ if $F$ is a number field and $L = k(t)$ otherwise. Viewing $F_\infty$ as a topological algebra over $H$ we identify it with the subalgebra of $F_A$, $\prod_{P \in S_\infty} F_P$.

Setting $X = F_\infty \times \prod_{P \notin S_\infty} R_P$, we observe that $F_A = X + F$ (see [7]). Let $\{\omega_1, \ldots, \omega_n\}$ be an integral basis of $F$ over $L$ with respect to $\Gamma$, where $\Gamma$ is the ring of rational integers if $F$ is a number field and otherwise $\Gamma = k[t]$. Finally we show that if $\zeta \in X$, then there exist $q, p_1, \ldots, p_n \in \Gamma$ such that $q \neq 0$ and $q\zeta - (p_1\omega_1 + \cdots + p_n\omega_n)$ has the property that

$$|(q\zeta - (p_1\omega_1 + \cdots + p_n\omega_n))_P|_P < 1 \quad \text{for } P \in S_\infty$$

and

$$|(q\zeta - (p_1\omega_1 + \cdots + p_n\omega_n))_P|_P \leqq 1 \quad \text{for } P \notin S_\infty,$$

i.e., $\zeta \in V(q^{-1}) + F$. Q.E.D.

Let $S$ be a finite set of prime divisors of $F$ such that $S \supseteq S_\infty$. We define a function $\varphi_S$ from $F$ to the nonnegative real numbers given by $\varphi_S(x) = \prod_{P \in S} |x|_P$. We note that, in view of (1), $\varphi_S$ is integral valued when restricted to $O_S$. Further in the case when $F$ is a number field and $S = S_\infty$, then, for all $x \in F$, $\varphi_S(x) = |N_{F/Q}(x)|$. Also when $F$ is a function field, then for any choice of $S \supseteq S_\infty$, there exist $y \in F - k$ such that $O_S$ is the integral closure of $k[y]$ in $F$ and, for all $x \in F$, $\varphi_S(x) = |N_{F/k(y)}(x)|_\infty$, where $|\ |_\infty$ is the valuation associated to the pole divisor of $y$ in $k(y)$ and normalized as above.

THEOREM 2. *There exists a finite set $S$ of prime divisors of $F$ such that $S \supseteq S_\infty$ and $O_S$ is euclidean with respect to the map $\varphi_S$.*

INDICATION OF PROOF. By Theorem 1, $F_A = \bigcup_{x \in F^*} (V(x) + F)$. Now by compactness of $F_A/F$ (see [7]) and the fact that $V(x)$ is open in $F_A$ for every $x \in F$, there exist $x_1, \ldots, x_r \in F^*$ such that

$$F_A = \bigcup_{i=1}^{r} (V(x_i) + F).$$

Finally we show that if $S = \{P | P \in S_\infty$ or there exist $i_0, 1 \leqq i_0 \leqq r$ such that $|x_{i_0}|_P \neq 1\}$, then $S$ is a finite set, $S \supseteq S_\infty$ and $O_S$ is euclidean with respect to $\varphi_S$.

## REFERENCES


1. J. V. Armitage, *Euclid's algorithm in certain algebraic function fields*, Proc. London Math. Soc. (3) **7** (1957), 498–509. MR **20** #30.
2. ———, *Corrigendum and Addendum*: "*Euclid's algorithm in algebraic function fields*", J. London Math. Soc. **43** (1968), 171–172. MR **37** #2727.
3. ———, *On unique factorization in algebraic function fields*, Illinois J. Math. **11** (1967), 280–283; Appendix, Illinois J. Math. **12** (1968), 5–6. MR **35** #172; MR **36** #5113.
4. ———, *Euclid's algorithm in algebraic function fields*. II, Acta Arith. **18** (1971), 337–348.
5. M. Madan and C. Queen, *Euclidean function fields*, J. Reine Angew. Math. (submitted).
6. P. Samuel, *About Euclidean rings*, J. Algebra **19** (1971), 282–301. MR **43** #6190.
7. A. Weil, *Basic number theory*, Die Grundlehren der math. Wissenschaften, Band 144, Springer-Verlag, New York, 1967. MR **38** #3244.


DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015